数理解析研究所講究録
1197 巻 2001 年 1-53

A Study of the Relativistic Euler Equation

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1 Introduction

In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

$$\frac{\partial}{\partial t}\frac{\rho+Pu^2/c^4}{1-u^2/c^2}+\frac{\partial}{\partial x}\frac{(\rho+P/c^2)u}{1-u^2/c^2}=0,$$
$$\frac{\partial}{\partial t}\frac{(\rho+P/c^2)u}{1-u^2/c^2}+\frac{\partial}{\partial x}\frac{P+\rho u^2}{1-u^2/c^2}=0, \tag{1.1}$$

$$\rho|_{t=0}=\rho_0(x), \qquad u|_{t=0}=u_0(x). \tag{1.2}$$

Here c is a positive constant, the speed of light, and P is a given function of ρ. The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When $c\to\infty$, (1.1) tends to the usual Euler equation of gas dynamics

$$\rho_t+(\rho u)_x=0,$$
$$(\rho u)_t+(P+\rho u^2)_x=0. \tag{1.3}$$

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume $P=\sigma^2\rho$, where σ is a positive constant $< c$. Under this assumption, they showed that if the initial data $\rho_0(x)$ and $u_0(x)$ satisfy

$$T.V.\log\rho_0<\infty, \qquad T.V.\log\frac{c+u_0}{c-u_0}<\infty,$$

then there exists a global weak solution to the Cauchy problem (1.1)(1.2). The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

$$P = Kc^5 f(y), \qquad \rho = Kc^3 g(y)$$
$$f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq,$$
$$g(y) = 3\int_0^y q^2\sqrt{1+q^2} dq.$$

For this equation of state, we have $P \sim \frac{c^2}{3}\rho$ as $\rho \to \infty$ but $P \sim \frac{1}{5}K^{2/3}\rho^{5/3}$ as $\rho \to 0$. So we assume the following properties of the function $P(\rho)$:

(A):

$$P(\rho) > 0, \qquad 0 < dP/d\rho < c^2, \qquad 0 < d^2P/d\rho^2$$

for $\rho > 0$, and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}/c^2]_1)$$

as $\rho \to 0$. Here A and γ are positive constants and

$$\gamma = 1 + \frac{2}{2N+1},$$

N being a positive integer, and $[X]_1$ denotes a convergent power series of the form $\sum_{k\geq 1} a_k X^k$.

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data $\rho_0(x)$, $u_0(x)$ satisfy

$$0 \leq \rho_0(x) \leq M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq M_0.$$

A weak solution of (1.1)(1.2) is defined as follows.

We write

$$\begin{aligned} E &= \frac{\rho + Pu^2/c^4}{1-u^2/c^2}, \\ F &= \frac{(\rho + P/c^2)u}{1-u^2/c^2}, \\ G &= \frac{P+\rho u^2}{1-u^2/c^2}, \\ U &= (E,F)^T, \qquad f(U) = (F,G)^T. \end{aligned}$$

Then (1.1) can be written as

$$U_t + f(U)_x = 0.$$

Let us denote by $U_0(x)$ the initial data. Then a weak solution $U(t,x)$ is a bounded measurable function which satisfies

$$\int\int (U\Phi_t + f(U)\Phi_x)dxdt + \int U_0(x)\Phi(0,x)dx = 0$$

for any test function $\Phi \in C_0^\infty([0,+\infty) \times R)$.

2 Riemann problems

The Riemann problem is the problem to the special initial data of the form

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases}$$

In order to slolve this we introduce the Riemann invariants

$$w = x + y, \qquad z = x - y$$

where

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

Then (1.1) is diagonarized as

$$w_t + \lambda_2 w_x = 0, \qquad z_t + \lambda_1 z_x = 0,$$

where

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \qquad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}.$$

the possible states $U = U_R$ connected to U_L on the right by rarefaction waves are

$$R_1: \qquad w = w_L, z > z_L$$

and

$$R_2: \qquad w > w_L, z = z_L.$$

The Rankine Hugoniot jump condition

$$\sigma[U] = [f(U)],$$

where $[U] = U_R - U_L, [f(U)] = f(U_R) - f(U_L)$, gives the shock curve

$$\frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}.$$

Along this curve we have shocks

$$S_1: \qquad \rho_L < \rho_R, u_R < u_L,$$

$$S_2: \qquad \rho_R < \rho_L, u_R < u_L.$$

The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vaccuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form

$$\Sigma_B = \{(w, z) | -B \leq z \leq w \leq B\},$$

we have the following

Proposition 1 *If the initial data U_L, U_R belong to Σ_B for some large B, then the solution of the Riemann problem is confined to Σ_B.*

Moreover if we consider the image of Σ_B in the (E, F)-space, we have

Proposition 2 *The region Σ_B is convex in the (E, F)-plane.*

Proof. Let us consider the above hedge $F = F(E)$ which corresponds to $w = B, -B < z < B$. We have to show $d^2F/dE^2 < 0$. Along the hedge $w = B$, we have

$$u = c \tanh \frac{1}{c}(B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2)\frac{\sqrt{P'}}{\rho + P/c^2}.$$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2} = \lambda_1.$$

Differentiating once more we have

$$\frac{d^2F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^4}(\frac{P''}{2\sqrt{P'}} + (1 - \frac{P'}{c^2})\frac{\sqrt{P'}}{\rho + P/c^2}) < 0.$$

This was to be seen. QED.

From Proposition 2, we have

Proposition 3 *If $U(s), s \in [a, b]$, is confined to a region Σ_B, then the average*

$$\frac{1}{b-a}\int_a^b U(s)ds$$

belongs to Σ_B.

Let us look at the shock wave which connects the left state U_L to the right state U_R with the shock speed σ.

The right state U_R and σ are parametrized by $\rho = \rho_R$. Then we have the following fact, which will be used in Section 4.

Proposition 4 *Along*$S_1(\rho_L < \rho)$*, we have* $d\sigma/d\rho < 0$*, and along* $S_2(\rho < \rho_L)$ *we have* $d\sigma/d\rho > 0$.

Proof. Without loss of generality we can assume $u_L = 0$. Then $u = u_R$ is given by

$$u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},$$

where $[\rho] = \rho - \rho_L, [P] = P - P_L$. We have

$$\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.$$

By a direct but tedious computations, we have

$$\begin{aligned}
\frac{d\sigma}{d\rho} &= \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2u(\rho_L + P/c^2)^2(\rho + P_L/c^2)^2},\\
X &= (\rho + P_L/c^2)(\rho + P/c^2)P'[\rho] + \\
&+ (\rho + P_L/c^2)(-(\rho + P_L/c^2) + [P]/c^2)[P] + \\
&- (\rho_L + P/c^2)[P]^2/c^2.
\end{aligned}$$

Since $P'' > 0$ we know $[P] \le P'[\rho]$. Thus

$$\begin{aligned}
X &\ge (\rho + P_L/c^2)(\rho + P/c^2)[P] + \\
&+ (\rho + P_L/c^2)(-(\rho_L + P_L/c^2) + [P]/c^2)[P] + \\
&- (\rho_L + P/c^2)[P]^2/c^2 \\
&= [P]((\rho + P_L/c^2)([\rho] + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2).
\end{aligned}$$

But

$$1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.$$

Using this, itis easy to see $X > 0$ both when $[\rho] > 0$ and when $[\rho] < 0$. Since $u < 0$, this completes the proof. QED.

3 Entropies

A pair of functions η and q is called an entropy- entropy flux if it satisfies the equation

$$D_U q = D_U \eta . D_U f. \tag{3.1}$$

Using the Riemann invariants, we can write (3.1) as

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

By eliminating q from the equation, we get the following second order equation:

$$\frac{\partial^2 \eta}{\partial w \partial z} + Q(J\frac{\partial \eta}{\partial w} - \frac{1}{J}\frac{\partial \eta}{\partial z}) = 0, \tag{3.2}$$

where

$$\begin{aligned} Q &= \frac{1}{4\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P''), \\ J &= \frac{1 - \sqrt{P'}u/c^2}{1 + \sqrt{P'}u/c^2}. \end{aligned}$$

Since this equation tends to the Euler-Poisson-Darboux equation

$$\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w - z}(\frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z}) = 0 \tag{3.3}$$

as $c \to \infty$, we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when $c = \infty$ the kinetic energy

$$\eta = \frac{1}{2}\rho u^2 + \frac{P}{\gamma - 1} \tag{3.4}$$

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as $c \to \infty$. Let us look for an entropy-entropy flux of the form

$$\eta = H(\rho, u^2), \qquad q = Q(\rho, u^2)u.$$

Inserting this to the equation it is easy to find an entropy-entropy flux

$$\begin{aligned} \eta^* &= -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2(\frac{\rho + Pu^2/c^4}{1 - u^2/c^2}), && (3.5) \\ q^* &= (-\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2\frac{\rho + P/c^2}{1 - u^2/c^2})u, && (3.6) \\ \Psi &= exp(\int_1^\rho \frac{d\rho}{\rho + P/c^2} + K_0), && (3.7) \end{aligned}$$

where K_0 is determined so that η^* tends to the kinetic energy (3.4) as $c = \infty$. We call the entropy η^* defined by (3.5) the relativistic standard entropy. The important fact is

Proposition 5 *The Hessian $D_U^2\eta^*$ is positive definite. For any fixed B there is a positive constant k such that*

$$(\xi | D_U^2 \eta^*(U)\xi) \geq k|\xi|^2,$$

for any $U \in \Sigma_B$ and $\xi = (\xi_0, \xi_1)$ with $|\xi|^2 = \xi_0^2 + \xi_1^2$.

Proof. The proof is due to direct but tedious calculations. We note

$$\begin{aligned}
\frac{\partial\rho}{\partial E} &= \frac{1+u^2/c^2}{1-P'u^2/c^4},\\
\frac{\partial u}{\partial E} &= -\frac{(1+P'/c^2)(1-u^2/c^2)u}{(\rho+P/c^2)(1-P'u^2/c^4)},\\
\frac{\partial\rho}{\partial F} &= -\frac{2u/c^2}{1-P'u^2/c^4},\\
\frac{\partial u}{\partial F} &= \frac{(1-u^2/c^2)(1+P'u^2/c^4)}{(\rho+P/c^2)(1-P'u^2/c4)}.
\end{aligned}$$

Using these, we have

$$\begin{aligned}
\frac{\partial\eta^*}{\partial E} &= -\frac{\Psi}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}}+c^2,\\
\frac{\partial\eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}},\\
\frac{\partial^2\eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(P'+2P'u^2/c^2+u^2),\\
\frac{\partial^2\eta^*}{\partial E\partial F} &= \frac{-\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(2P'/c^2+1+P'u^2/c^4)u,\\
\frac{\partial^2\eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(1+3P'u^2/c^4).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(\xi|D_U^2\eta^*\xi) &= \eta^*_{EE}\xi_0^2+2\eta^*_{EF}\xi_0\xi_1+\eta^*_{FF}\xi_1^2\\
&= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}Z,\\
Z &= (P'+2P'u^2/c^2+u^2)\xi_0^2-2(2P'/c^2+1+P'u^2/c^4)u\xi_0\xi_1+\\
&+ (1+3P'u^2/c^4)\xi_1^2\\
&\geq \frac{2P'(1-u^2/c^2)^2(1-P'u^2/c^4)}{A+C+\sqrt{(A-C)^2+4B^2}}(\xi_0^2+\xi_1^2),\\
A &= P'+2P'u^2/c^2+u^2,\\
B &= (2P'/c^2+1+P'u^2/c^4)u,\\
C &= 1+3P'u^2/c^4.
\end{aligned}$$

This completes the proof. QED.

4 Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.

Suppose that the initial data $U_0(x)$ is confined to an invariant region Σ_B. Put $\Lambda_0 = \sup\{|\lambda_j(U)||j = 1, 2, U \in \Sigma_B\}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approxomate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x)\chi_{[-1/\Delta, 1/\Delta]}.$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x}\int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x)dx$$

for $2j\Delta x < x \le (2j+2)\Delta x$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \le t < \Delta t$. Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \le t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x)dx$$

for $2j\Delta < x \le (2j+2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t, x)$ for $n\Delta t \le t < (n+1)\Delta t$.

By Proposition 1 and 3, it is inductively guaranteed that U^Δ remains in Σ_B, say,

Proposition 6 *The approximate solution $U^\Delta(t, x)$ satisfies $U^\Delta(t, x) \in \Sigma_B$, therefore,*

$$0 \le \rho^\Delta(t, x) \le M, \qquad |\frac{c}{2}\log\frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)}| \le M.$$

Moreover we shall prove

Proposition 7 *For any test function Φ it holds that*

$$\int\int(\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0, x)U_0^\Delta(x)dx = O(\Delta^{1/2}).$$

In order to prove Proposition 7, we prepare

Proposition 8 *For any shock wave from U_L to U_R with the shock speed σ and for any convex entropy η, we have*

$$\sigma[\eta] - [q] \ge 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.

Proof. The right state of shocks can be parametrized by $\rho = \rho_R$. Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see $dQ/d\rho \geq 0$ along $S_1 : [\rho] > 0$ and $dQ/d\rho \leq 0$ along $S_2 : [\rho] < 0$. Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\begin{aligned}\frac{dQ}{d\rho} &= \frac{d\sigma}{d\rho}([\eta] - D_U\eta(U).[U]) \\ &= -\frac{d\sigma}{d\rho}\int_0^1 \theta(U - U_L|D_U^2\eta(U_L + \theta(U - U_L).(U - U_L))d\theta.\end{aligned}$$

We supposed $D_U^2\eta \geq 0$. By Proposition 4, we know $d\sigma/d\rho < 0$ on S_1 and $d\sigma/d\rho > 0$ on S_2. QED.

Proof of Proposition 7.

We fix T to consider U^Δ on $0 \leq t \leq T$. First we shall show

$$\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \leq C. \qquad (4.1)$$

Let us consider the standard entropy η^*. Then we have

$$\begin{aligned}0 &= \int \eta^*(U(T,x))dx - \int \eta^*(U(0,x))dx + L + \Sigma, \\ L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}(\eta^*(U(n\Delta t - 0, x)) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta)))dx, \\ \Sigma &= \int_0^T \sum_{shocks}(\sigma[\eta^*] - [q^*])dt.\end{aligned}$$

We write $U_0 = U(n\Delta t + 0, (2j+1)\Delta), U_1 = U(n\Delta t - 0, x)$. Since

$$U_0 = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$\begin{aligned}L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}\int_0^1 (1-\theta)(U_1 - U_0|D_U^2\eta^*(U_0 + \theta(U_1 - U_0)).(U_1 - U_0))d\theta dx \\ &\geq 0.\end{aligned}$$

On the other hand we have $\Sigma \geq 0$ from Proposition 8. Thus $L \leq C, \Sigma \leq C$. But from Proposition 5, we have $D_U^2\eta^* \geq k$. Therefore

$$C \geq L \geq \frac{k}{2}\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}|U_1 - U_0|^2 dx.$$

Thus we get (4.1).

Now let us consider a test function Φ. Put

$$J = \int\int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0,x) U_0^\Delta dx.$$

Since U^Δ is a weak solution on each time strip $n\Delta t < t < (n+1)\Delta t$, we have

$$\begin{aligned} J &= \sum_n \int \Phi(n\Delta t, x)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx \\ &= J_1 + J_2, \\ J_1 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx, \\ J_2 &= \sum_{j.n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t,x) - \Phi(n\Delta t, j\Delta))(U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx. \end{aligned}$$

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x) dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$\begin{aligned} |J_2| &\leq C\Delta^{1/2} ||\Phi||_{C^1} (\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx)^{1/2} \\ &\leq C'\Delta^{1/2}. \end{aligned}$$

Here we have used $T/\Delta t = O(1/\Delta)$. QED.

Summing up, we have the following theorem.

Theorem 1 *The approximate solution $U^\Delta(t,x)$ satisfies*

$$0 \leq \rho^\Delta(t,x) \leq M, \qquad |\frac{c}{2} \log \frac{c + u^\Delta(t,x)}{c - u^\Delta(t,x)}| \leq M$$

and

$$\int\int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0,x) U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function Φ.

We expect that U^Δ tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s) ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3) , ϕ being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).

5 Remark

We note that

$$\begin{aligned}
\lambda_2 - \lambda_1 &= \frac{\sqrt{P'}(1-u^2/c^2)}{1-u^2P'/c^4} > 0, \\
\frac{\partial \lambda_1}{\partial z} &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) > 0, \\
\frac{\partial \lambda_2}{\partial w} &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) > 0
\end{aligned}$$

for $\rho > 0$ and $|u| < c$.

This says that the system is strictly hyperbolic and genuinely nonlinear on $\rho > 0$. Therefore the Glimm's theory can be applied if

$$||U_0(x) - U^*||_{L^\infty} + T.V.U_0$$

is sufficiently small, where U^* is a constant state such that $\rho^* > 0, |u^*| < c$. But the vaccum may not be covered by this application of the general theorem.

6 Generalized Darboux formula

In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho+P/c^2}d\rho.$$

Then the relativistic Euler-Poisson-Darboux equation is

$$(EPD) \qquad \eta_{xx} - \eta_{yy} + A(x,y)\eta_y + B(x,y)\eta_x = 0,$$

where

$$\begin{aligned}
A(x,y) &= \frac{1}{\sqrt{P'}}(1-\frac{P'}{c^2}-\frac{\rho+P/c^2}{2P'}P'')\frac{1+P'u^2/c^4}{1-P'u^2/c^4}, \\
B(x,y) &= -\frac{2u/c^2}{1-P'u^2/c^4}(1-\frac{P'}{c^2}-\frac{\rho+P/c^2}{2P'}P'').
\end{aligned}$$

The coefficients A and B are of the form

$$\begin{aligned}
A &= \frac{2N}{y} + a, \qquad a = \frac{y}{c^2}(a_0 + [x^2/c^2, y^2/c^2]_1), \\
B &= -\frac{4N}{N+1}\frac{x}{c^2}(1+[x^2/c^2, y^2/c^2]_1),
\end{aligned}$$

where $[X,Y]_1$ denotes a convergent power series $\sum_{j+k\geq 1} c_{jk}X^jY^k$. In order to remove the singularity in A, we use the trick of Weinstein [7]. We introduce the sequence of variables $\eta_j, j = 0, 1, , ..., N$ by

$$\frac{\partial \eta_j}{\partial y} = y\eta_{j+1},$$

or

$$\eta_j(x,y) = I\eta_{j+1}(x,y) = \int_0^y Y\eta_{j+1}(x,Y)dY,$$

where $\eta_0 = \eta$. The sequence of formal integro-differential operators L_j is defined by

$$\begin{aligned} L_jV &= V_{xx} - V_{yy} + (\frac{2(N-j)}{y} + a)V_y + BV_x + \\ &+ j\tilde{a}V + \sum_{k=1}^{j} F_{jk}I^kV_x + \sum_{k=1}^{j} H_{jk}I^kV, \end{aligned}$$

where

$$\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0.$$

The coefficients F_{jk} and H_{jk} are determined inductively by

$$F_{j+1,k} = \begin{cases} F_{j1} + \frac{1}{y}\frac{\partial B}{\partial y} & \text{if } k = 1 \\ F_{jk} + \frac{1}{y}\frac{\partial}{\partial y}F_{j,k-1} & \text{if } k \geq 2 \end{cases}$$

$$H_{j+1,k} = \begin{cases} H_{j1} + j\frac{1}{y}\frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1 \\ H_{jk} + \frac{1}{y}\frac{\partial}{\partial y}H_{j.k-1} & \text{if } k \geq 2 \end{cases}$$

It is easy to see that F_{jk} are of the form $\frac{x}{c^2}[x^2/c^2, y^2/c^2]_0$ and H_{jk} are of the form $\frac{1}{c^2}[x^2/c^2, y^2/c^2]_0$. By the definition we have formally

$$\frac{1}{y}\frac{\partial}{\partial y}(L_j\eta_j) = L_{j+1}\eta_{j+1}.$$

Now we consider the equation $L_NV = 0$ for $V = \eta_N$ with the initial conditions

$$V = 0, \qquad V_y = 2^{N+1}N!\phi(x), \qquad \text{at } y = 0.$$

The problem is

$$(Q) \qquad \begin{aligned} &V_{yy} - V_{xx} = aV_y + BV_x + N\tilde{a}V + \\ &+\sum_{k=1}^{N} F_kI^kV_x + \sum_{k=1}^{N} H_kI^kV, \\ &V = 0, \qquad V_y = 2^{N+1}N!\phi(x) \qquad \text{at } y = 0. \end{aligned}$$

Proposition 9 *If $\phi \in C^1(R)$, then the problem (Q) admits a unique solution V in $C^2(R \times [0,\infty))$.*

Proof. Let us denote by $H(x,y,V)$ the right hand side of the equation $L_N = 0$. Then (Q) is transformed to the integral equation

$$V(x,y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X,Y,V)dXdY.$$

We can solve this integral equation by the iteration

$$\begin{aligned} V_0(x,y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi, \\ V^{n+1}(x,y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X,Y,V^n)dXdY. \end{aligned}$$

Fixing L arbitrarily, we consider $|x| \le L$. Then it is easy to get the estimates

$$|V^{n+1}(x,y) - V^n(x,y)| \le \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

Therefore V^n tends to a limit V uniformly on $|x| \le L, 0 \le y \le L$. The limit is the unique solution of (Q). QED.

Now we put

$$\eta_N = V, \qquad \eta_{N-k} = I\eta_{N-k+1}.$$

Since η_{N-k} and its derivatives of order ≤ 2 all vanish on $y=0$ for $k \ge 1$, we see $\eta = \eta_0$ gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution V of (Q).

Proposition 10 *There is a C^{N+2}-function $G(x,y,\xi)$ of $|x| < \infty, y \ge 0, x-y \le \xi \le x+y$ such that the solution V of (Q) satisfies*

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi)\phi(\xi)d\xi. \tag{6.1}$$

Moreover

$$\begin{aligned} G &= 2^N N! + O(y/c^2), \\ \partial_x^{p_1}\partial_\xi^{p_2}\partial_y^{p_3} G &= O(1/c^2) \qquad \text{for } 1 \le p_1+p_2+p_3 \le N+2 \end{aligned}$$

Proof. We consider the approximate solution $V^n(x,y)$ which appeared in the iteration of the proof of Proposition 9. By writing H as

$$H = (aV)_y + (BV)_x + bV + \sum (F_k I^k V)_x + \sum \tilde{H}_k I^k V,$$

where

$$\begin{aligned} b &= N\tilde{a} - a_y - B_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \\ \tilde{H}_k &= H_k - (F_k)_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \end{aligned}$$

it is easy to see inductively that there is a kernel $G^n(x,y,\xi)$ such that

$$V^n(x,y) = \int_{x-y}^{x+y} G^n(x,y,\xi)\phi(\xi)d\xi.$$

In fact $G^0 = 2$ and G^n are determined inductively by the formula

$$\begin{aligned} G^{n+1} &= 2 + \frac{1}{2}(G_I^n + G_{II}^n + G_{III}^n + \sum G_{IVk}^n + \sum G_{Vk}^n), \\ G_I &= \int_{(-x+y+\xi)/2}^{y} a(x-y+Y,Y)G(x-y+Y,Y,\xi)dY + \\ &+ \int_{(x+y-\xi)/2}^{y} a(x+y-Y,Y)G(x+y-Y,Y,\xi)dY, \\ G_{II} &= \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)G(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)G(x-y+Y,Y,\xi)dY, \\ G_{III} &= \int\int_{D(x,y,\xi)} b(X,Y)G(X,Y,\xi)dXdY, \end{aligned}$$

where

$$\begin{aligned} D(x,y,\xi) &= \{(X,Y) | X - Y \le \xi \le X + Y, x - y + Y \le X \le x + y - Y, 0 \le Y \le y\}, \\ G_{IVk} &= \int_{(x-y+\xi)/2}^{y} F_k(x+y-Y,Y)J^k G(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} F_k(x-y+Y,Y)J^k G(x-y+Y,Y,\xi)dY, \end{aligned}$$

where

$$JG(x,y,\xi) = \int_{|x-\xi|}^{y} YG(x,Y,\xi)dY,$$

and

$$G_{Vk} = \int\int_{D(x,y,\xi)} \tilde{H}_k(X,Y)J^k G(X,Y,\xi)dXdY.$$

It is easy to see inductively that

$$|G^{n+1}(x,y,\xi) - G^n(x,y,\xi)| \le \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

therefore G^n converges to a limit G uniformly and (6.1) holds. Moreover we can differentiate G^{n+1} supposing that G^n is differentiable. In fact we have

$$\begin{aligned}
G_{I,x} &= \frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)\\
&- \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \int_{(-x+y+\xi)/2}^{y}(aG)_x(x-y+Y,Y,\xi)dY\\
&+ \int_{(x+y-\xi)/2}^{y}(aG)_x(x-Y+Y,Y,\xi)dY,\\
G_{I,\xi} &= -\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \int_{(-x+y+\xi)/2}^{y}aG_\xi(x-y+Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}aG_\xi(x+y-Y,Y,\xi)dY,\\
G_{I,y} &= -\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&- \frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ 2aG(x,y,\xi)+\\
&- \int_{(-x+y+\xi)/2}^{y}(aG)_x(x-y+Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}(aG)_x(x+y-Y,Y,\xi)dY;\\
G_{II,x} &= -\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&- \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY+\\
&- \int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY,\\
G_{II,\xi} &= \frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \int_{(x+y-\xi)/2}^{y}BG_\xi(x+y-Y,Y,\xi)dY+\\
&- \int_{(-x+y+\xi)/2}^{y}BG_\xi(x-y+Y,Y,\xi)dY,
\end{aligned}$$

$$
\begin{aligned}
G_{II,y} &= -\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY;\\
G_{III,x} &= \int_{(x+y-\xi)/2}^{y}bG(x+y-Y,Y,\xi)dY-\int_{(-x+y+\xi)/2}^{y}bG(x-y+Y,Y,\xi)dY,\\
G_{III,\xi} &= \int_0^{(x+y-\xi)/2}bG(\xi+Y,Y,\xi)dY+\int_0^{(-x+y+\xi)/2}bG(\xi-Y,Y,\xi)dY+\\
&+ \int\int_{D(x,y,\xi)}bG(X,Y,\xi)dXdY,\\
G_{III,y} &= \int_{(x+y-\xi)/2}^{y}bG(x+y-Y,Y,\xi)dY+\int_{(-x+y+\xi)/2}^{y}bG(x-y+Y,Y,\xi)dY;
\end{aligned}
$$

and the derivatives of G_{IVk} are similar to G_{II} and the derivatives of G_{IVk} are similar to G_{III}. Then it is easy to see inductively that

$$
|G_x^{n+1}-G_x^n|+|G_\xi^{n+1}-G_\xi^n|+|G_y^{n+1}-G_y^n|\le\frac{M^ny^n}{n!}.
$$

Thus the limit G is differentiable. In a smilar manner we see

$$
\begin{aligned}
|G_{xx}^{n+1}-G_{xx}^n| &+ |G_{x\xi}^{n+1}-G_{x\xi}^n|+|G_{xy}^{n+1}-G_{xy}^n|+\\
&+ |G_{\xi\xi}^{n+1}-G_{\xi\xi}^n|+|G_{\xi y}^{n+1}-G_{\xi y}^n|+|G_{yy}^{n+1}-G_{yy}^n|\le\\
&\le \frac{M^{n-1}y^{n-1}}{(n-1)!}.
\end{aligned}
$$

Thus G is twice continuously differentiable. In a similar manner we see that G is $N+2$-times continuously differentiable. The rough estimates stated in the propositions is obvious since the coefficients are all of $O(1/c^2)$. QED.

The solution η_{N-k} enjoyes an integral representation

$$
\eta_{N-k}=\int_{x-y}^{x+y}K_{N-k}(x,y,\xi)\phi(\xi)d\xi,
$$

where

$$
K_{N-k}(x,y,\xi)=JK_{N-k+1}(x,y,\xi)=J^kG(x,y,\xi).
$$

So the solution η of the relativistic Euler-Poisson-Darboux equation is given by

$$
\eta(x,y)=\int_{x-y}^{x+y}K(x,y,\xi)\phi(\xi)d\xi,
$$

where

$$K(x,y,\xi) = J^N G(x,y,\xi).$$

By induction we see

$$J^k G(x,y,\xi) = \frac{2^N N!}{2^k k!}(y^2 - (x-\xi)^2)^k (1 + O(y/c^2)).$$

Thus we have

Proposition 11 *There is a kernal $K(x,y,\xi)$ which is of C^{N+2}-class in $|x| < \infty, 0 \le y, x-y \le \xi \le x+y$ such that*

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi$$

gives a solution of the relativistic Euler- Poisson-Darboux equation for any smooth ϕ. Moreover

$$K(x,y,\xi) = (y^2 - (x-\xi)^2)^N (1 + O(y/c^2)).$$

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

Proposition 12 *We have*

$$G_y = O(y/c^2).$$

Proof. Since $a = O(y/c^2)$, it is clear that $G_{I,y} = O(y/c^2)$. Next we see

$$G_{II,y} = -B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2)) + O(y/c^2).$$

On the other hand we can write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2)$$

and

$$\frac{x+y+\xi}{2} = x + \frac{y+Z}{2}, \qquad \frac{x-y+\xi}{2} = x + \frac{-y+Z}{2}, \qquad Z = \xi - x.$$

Therefore we see $G_{II,y} = O(y/c^2)$. It is clear that $G_{III,y} = O(y/c^2)$ and $G_{IVk,y}, G_{Vk,y} = O(y^2/c^2)$. QED.

Proposition 13 *We have*

$$G = 2^N N! + \frac{1}{c^2}C_0(x,c)(\xi - x) + O(y^2/c^2),$$

where $C_0(x,c)$ is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. It is clear that $G_I = O(y^2/c^2)$ since $a = O(y/c^2)$. Next we see

$$G_{II} = 2^N N! \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - 2^N N1 \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY + O(y^2/c^2),$$

since $G = 2^N N! + O(y/c^2)$. If we write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2), \qquad Z = \xi - x$$

, then we see

$$\begin{aligned}
\int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY \quad &- \quad \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY = \\
&= \quad \frac{1}{c^2}(\int_{x}^{x+\frac{y+Z}{2}} B_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} B_0(s)ds) + O(y^2/c^2) \\
&= \quad \frac{1}{c^2}B_0(x)Z + O(y^2/c^2).
\end{aligned}$$

Note $|Z| \leq y$. It is clear that $G_{III}, G_{IVk}, G_{Vk} = O(y^2/c^2)$. QED.

Proposition 14 *We have*

$$G_x + G_\xi = O(y/c^2).$$

Proof. First we see

$$\begin{aligned}
G_{I,x} + G_{I,\xi} &= \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)(x-y+Y,Y,\xi)dY + \\
&+ \int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x+y-Y,Y,\xi)dY \\
&= O(y^2/c^2),
\end{aligned}$$

since $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned}
G_{II,x} + G_{II,\xi} &= \int_{(x+y-\xi)/2}^{y} ((BG)_x) + BG_\xi)(x+y-Y,Y,\xi)dY + \\
&- \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY \\
&= O(y/c^2).
\end{aligned}$$

It is clear that $G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)$. $G_{IVk,x} + G_{IVk,\xi}$ is estimated in a similar manner as $G_{II,x} + G_{II,\xi}$. QED.

Proposition 15 *We have*

$$(G_x + G_\xi)_y = O(y/c^2).$$

Proof. First we see

$$
\begin{aligned}
(G_{I,x}+G_{I,\xi})_y &= 2((aG)_x+aG_\xi)(x,y,\xi)+\\
&- \frac{1}{2}((aG)_x+aG_\xi)((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&- \frac{1}{2}((aG)_x+aG_\xi)((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&- \int_{(-x+y+\xi)/2}^{y}((aG)_x+aG_\xi)_x(x-y+Y,Y,\xi)dY+\\
&+ \int_{((x+y-\xi)/2}^{y}((aG)_x+aG_\xi)_x(x+y-Y,Y,\xi)dY\\
&= O(y/c^2),
\end{aligned}
$$

since $a, a_x = O(y/c^2)$. Next we see

$$
\begin{aligned}
(G_{II,x}+G_{II,\xi})_y &= -\frac{1}{2}((BG)_x+BG_\xi)((x+y+\xi)/2,(x+y-\xi)/2,\xi)+\\
&+ \frac{1}{2}((BG)_x+BG_\xi)((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&+ \int_{(x+y-\xi)/2}^{y}((BG)_x+BG_\xi)_x(x+y-Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}((BG)_x+BG_\xi)_x(x-y+Y,Y,\xi)dY\\
&= 2^{N-1}N!B_x((x-y+\xi)/2,(-x+y+\xi)/2)\\
&- 2^{N-1}N!B_x((x+y+\xi)/2,(x+y-\xi)/2)+\\
&+ O(y/c^2),
\end{aligned}
$$

since $G = 2^N N! + O(y/c^2)$ and $G_x + G_\xi = O(y/c^2)$. But

$$B_x = \frac{1}{c^2}B_0'(x) + O(y^2/c^2)$$

and

$$
\begin{aligned}
B_x((x-y+\xi)/2,(-x+y+\xi)/2) &- B_x((x+y+\xi)/2,(x+y-\xi)/2) =\\
&= \frac{1}{c^2}B_0''(x)(-y)+O(y^2/c^2)\\
&= O(y/c^2).
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
(G_{III,x}+G_{III,\xi})_y &= \int_{(x+y-\xi)/2}^{y}((bG)_x+bG_\xi)(x+y-Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}((bG)_x+bG_\xi)(x-y+Y,Y,\xi)dY\\
&= O(y/c^2).
\end{aligned}
$$

Similarly we can estimate $(G_{IVk,x}+G_{IVk,\xi})_y, (G_{Vk,x}+G_{Vk,\xi})_y$ bearing in mind that $(JG)_x+(JG)_\xi = J(G_x+G_\xi)$. QED.

Proposition 16 *We have*

$$G_x + G_\xi = \frac{1}{c^2}C_1(x,c)(\xi - x) + O(y^2/c^2),$$

where $C_1(x,c)$ *is a function of the form*

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. We already observed that $G_{Ix}+G_{I\xi} = O(y^2/c^2)$. Next we look at

$$\begin{aligned} G_{II,x} \quad + \quad G_{II,\xi} &= \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x+y-Y,Y,\xi)dY + \\ &- \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY \\ &= 2^N N! \int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY + \\ &+ O(y^2/c^2), \end{aligned}$$

since $G = 2 + O(y/c^2)$ and $G_x + G_\xi = O(y/c^2)$. Bearing in mind that $B_y = O(y/c^2)$, we see

$$\begin{aligned} \int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY &- \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY = \\ = -\int_{(x+y-\xi)/2}^{y} (-B_x+B_y)(x+y-Y,Y)dY &- \int_{(-x+y+\xi)/2}^{y} (B_x+B_y)(x-y+Y,Y)dY \\ &+ O(y^2/c^2) \\ = -2B(x,y) &+ B((x+y+\xi)/2,(x+y-\xi)/2) + \\ +B((x-y+\xi)/2,(-x+y+\xi)/2) &+ O(y^2/c^2) \\ = \frac{1}{c^2}(-2B_0(x) + B_0(x+\frac{y+Z}{2}) &+ B_0(x+\frac{-y+Z}{2})) + O(y^2/c^2) \\ &= \frac{1}{c^2}B_0'(x)Z + O(y^2/c^2). \end{aligned}$$

Next we look at

$$\begin{aligned} G_{III,x}+G_{III,\xi} &= \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY + \\ &+ \int_0^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY - \int_0^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY + \\ &+ \int\int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY. \end{aligned}$$

Putting
$$b(x,y)=\frac{1}{c^2}b_0(x)+O(y^2/c^2),$$
we see

$$\begin{aligned}
G_{III,x}+G_{III,\xi} &= 2^N N!(\int_x^{x+\frac{y+Z}{2}} b_0(s)ds-\int_{x+\frac{-y+Z}{2}}^{x} b_0(s)ds+\\
&+\int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s)ds-\int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s)ds)+O(y^2/c^2)\\
&= \frac{2^N N!}{c^2}b_0(x)(\frac{y+Z}{2}-\frac{y-Z}{2}+\frac{y-Z}{2}-\frac{y+Z}{2})+O(y^2/c^2)\\
&= O(y^2/c^2).
\end{aligned}$$

$G_{IVk,x}+G_{IVk,\xi}$ can be estimated in a similer manner as $G_{II,x}+G_{II,\xi}$. Finally $G_{Vk,x}, G_{Vk,\xi}=O(y^3/c^2)$ since $J^kG=O(y^2/c^2)$ for $k\geq 1$. QED.

Proposition 17 *We have*
$$(G_x+G_\xi)_x+(G_x+G_\xi)_\xi=O(y/c^2).$$

Proof. First we see

$$\begin{aligned}
(G_{I,x}+G_{I,\xi})_x &+ (G_{I,x}+G_{I,\xi})_\xi=\\
&= \int_{(-x+y+\xi)/2}^{y}((aG)_{xx}+2(aG_\xi)_x+aG_{\xi\xi})(x-y+Y,Y,\xi)dY+\\
&+ \int_{(x+y-\xi)/2}^{y}((aG)_{xx}+2(aG_\xi)_x+aG_{\xi\xi})(x+y-Y,Y,\xi)dY\\
&= O(y^2/c^2),
\end{aligned}$$

since $a, a_x, a_{xx}=O(y/c^2)$. Next

$$\begin{aligned}
(G_{II,x} &+ G_{II,\xi})_x+(G_{II,x}+G_{II,\xi})_\xi=\\
&= \int_{(x+y-\xi)/2}^{y}(((BG)_x+BG_\xi)_x+((BG)_x+BG_\xi)_\xi)(x+y-Y,Y,\xi)dY+\\
&+ \int_{(-x+y+\xi)/2}^{y}(((BG)_x+BG_\xi)_x+((BG)_x+BG_\xi)_\xi)(x+y-Y,Y,\xi)dY\\
&= O(y/c^2).
\end{aligned}$$

It is easy to see
$$(G_{III,x}+G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi=O(y/c^2).$$

The estimates of G_{IVk} and G_{Vk} can be seen similarly. QED.

Proposition 18 *We have*

$$(G_x+G_\xi)_x+(G_x+G_\xi)_\xi=\frac{1}{c^2}C_2(x,c)(\xi-x)+O(y^2/c^2),$$

where $C_2(x,c)$ is a function of the form

$$[x^2/c^2]_0+\frac{x}{c^2}[x^2/c^2]_0.$$

Proof. We already observed that

$$(G_{I,x}+G_{I,\xi})_x+(G_{I,x}+G_{I,\xi})_\xi=O(y^2/c^2).$$

Next, bearing in mind that $G_x+G_\xi=O(y/c^2)$ and $(G_x+G_\xi)_x+(G_x+G_\xi)_\xi=O(y/c^2)$, we see

$$\begin{aligned}(G_{II,x} &+ G_{II,\xi})_x+(G_{II,x}+G_{II,\xi})_\xi=\\ &= \int_{(x+y-\xi)/2}^{y}(B_{xx}G+2B_x(G_x+G_\xi)+\\ &+ B((G_x+G_\xi)_x+(G_x+G_\xi)_\xi))(x+y-Y,Y,\xi)dY+\\ &- \int_{(-x+y+\xi)/2}^{y}(B_{xx}G+2B_x(G_x+G_\xi)+\\ &+ B((G_x+G_\xi)_x+(G_x+G_\xi)_\xi)(x-y+Y,Y,\xi)dY\\ &= 2^N N!\int_{(x+y-\xi)/2}^{y}B_{xx}(x+y-Y,Y)dY-2^N N!\int_{(-x+y+\xi)/2}^{y}B_{xx}(x-y+Y,Y)dY+\\ &+ O(y^2/c^2).\end{aligned}$$

The same discussion to that of the proof of Proposition 16 can be applied by replacing B by B_x. Let us look at $(G_{III,x}+G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi$. Note that

$$\begin{aligned}(bG)_x+bG_\xi &= b_xG+b(G_x+G_\xi)\\ &= 2^N N!b_x+O(y/c^2),\\ bG &= 2^N N!b+O(y/c^2).\end{aligned}$$

Applying the discussion of the proof of Proposition 16 by replacing b by b_x, we see

$$\begin{aligned}(G_{III,x} &+ G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi=\\ &= 2^N N!(\int_{x+Z}^{x+\frac{y+Z}{2}}b_0(s)ds-\int_{x+\frac{-y+Z}{2}}^{x+Z}b_0(s)ds)+\\ &+ O(y^2/c^2)\\ &= -2^N N!b_0(x)Z+O(y^2/c^2).\end{aligned}$$

The estimates of G_{IVk}, G_{Vk} are paralell. QED.

Proposition 19 *We have*

$$G_\xi = \frac{1}{c^2}C_3(x,c) + O(y/c^2).$$

Proof. It is sufficient to note that

$$\begin{aligned} G_{II,\xi} &= 2^{N-1}N!(B((x+y+\xi)/2,(x+y-\xi)/2) + B((x-y+\xi)/2,(-x+y+\xi)/2)) + \\ &+ O(y/c^2) \\ &= \frac{2^{N-1}N!}{c^2}(B_0(x+\frac{y+Z}{2}) + B_0(x+\frac{-y+Z}{2})) + O(y/c^2) \\ &= \frac{2^N N!}{c^2}B_0(x) + O(y/c^2). \end{aligned}$$

QED.

Proposition 20 *We have*

$$(G_x + G_\xi)_\xi = \frac{1}{c^2}C_4(x,c) + O(y/c^2).$$

Proof. We see

$$(G_{I,x} + G_{I,\xi})_\xi = O(y/c^2)$$

by $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned} (G_{II,x} &+ G_{II,\xi})_\xi = \\ &= 2^{N-1}N!(B_x((x+y+\xi)/2,(x+y-\xi)/2) + \\ &+ B_x((x-y+\xi)/2,(-x+y+\xi)/2)) + O(y/c^2) \\ &= \frac{2^N N!}{c^2}B_0'(x) + O(y/c^2). \end{aligned}$$

And we see

$$\begin{aligned} (G_{III,x} &+ G_{III,\xi})_\xi = \\ &= 2^N N! b((x-y+\xi)/2,(-x+y+\xi)/2) + O(y/c^2) \\ &= \frac{2^N N!}{c^2}b_0(x) + O(y/c^2). \end{aligned}$$

Other terms can be estimated similarly. QED.

7 Estimates of the derivatives of entropies

Let us consider the entropy η generated by ϕ of C^3-class, that is,

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi.$$

In this section we will find estimates of the derivatives of η with respect to E, F. As auxiliary variables we introduce

$$R = y^{2N+1}, \qquad M = xy^{2N+1}. \tag{7.1}$$

We are going to prove the following

Proposition 21 *We have*

$$\begin{aligned}
\frac{\partial \eta}{\partial M} &= 2^{2N+1}\int_0^1 (s-s^2)^N D\phi(x+(2s-1)y)ds + O(y^2/c^2), & (7.2)\\
\frac{\partial \eta}{\partial R} &= 2^{2N+1}\int_0^1 (s-s^2)^N \phi ds + \\
&\quad 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds + \\
&\quad O(y^2/c^2), & (7.3)\\
\frac{\partial^2 \eta}{\partial M^2} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi ds + O(y^{-2N+1}/c^2), & (7.4)\\
\frac{\partial^2 \eta}{\partial R \partial M} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds + \\
&\quad O(y^{-2N+1}/c^2), & (7.5)\\
\frac{\partial^2 \eta}{\partial R^2} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N((-x+\frac{y}{2N+1}(2s-1))^2 + \\
&\quad \frac{4}{(2N+1)^2}s(1-s)y^2)D^2\phi(x+(2s-1)y)ds + O(y^{-1}/c^2) & (7.6)
\end{aligned}$$

Proof. We write

$$\eta = 2R^{\frac{1}{2N+1}}\int_0^1 K(\frac{M}{R}, R^{\frac{1}{2N+1}}, \frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})\phi(\frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})ds.$$

Differentiating η with respect to M, we have

$$\begin{aligned}
\frac{\partial \eta}{\partial M} &= (1)+(2),\\
(1) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds,\\
(2) &= 2R^{\frac{-2N}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)D\phi(x+(2s-1)y)ds.
\end{aligned}$$

Since $K(x,y,\xi) = J^N G(x,y,\xi)$,i.e.

$$K(x,y,\xi) = \int_{|x-\xi|}^y Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1}\cdots\int_{|x-\xi|}^{Y_2} Y_1 G(x,Y_1,\xi)dY_1\cdots Y_N,$$

by Proposition 16 we see

$$
\begin{aligned}
(K_x \quad + \quad & K_\xi)(x,y,x+(2s-1)y) \\
= \quad & \int_{|2s-1|y}^{y} Y_N \int_{|2s-1|y}^{Y_N} Y_{N-1} \cdots \int_{|2s-1|y}^{Y_2} Y_1 (G_x + G_\xi)(x, Y_1, x+(2s-1)y) dY_1 \cdots Y_N \\
= \quad & \frac{C_1(x,c)}{2^N N! c^2} y^{2N+1}(2s-1)(1-(2s-1)^2)^N + O(y^{2N+2}/c^2) \\
= \quad & -\frac{2^N C_1(x,c)}{(N+1)! c^2} y^{2N+1} \frac{d}{ds}(s-s^2)^{N+1} + O(y^{2N+2}/c^2).
\end{aligned}
$$

Therefore by integration by part we get

$$
\begin{aligned}
(1) \quad = \quad & R^{\frac{-2N}{2N+1}} y^{2N+2} \frac{2^{N+1} C_1(x,c)}{(N+1)! c^2} \int_0^1 (s-s^2)^{N+1} D\phi ds + O(y^2/c^2) \\
= \quad & O(y^2/c^2).
\end{aligned}
$$

By Proposition 13 we see

$$
\begin{aligned}
K(x,y,\xi) \quad = \quad & \int_{|x-\xi|}^{y} Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1} \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x, Y_1, \xi) dY_1 \cdots Y_N, \\
= \quad & 2^{2N}(s-s^2)^N y^{2N} + \frac{2^N C_0(x,c)}{N! c^2}(2s-1)(s-s^2)^N y^{2N+1} + O(y^{2N+2}/c^2).
\end{aligned}
$$

Therefore by integration by parts we get

$$
\begin{aligned}
(2) \quad = \quad & 2^{2N+1} R^{\frac{-2N}{2N+1}} y^{2N} \int_0^1 (s(1-s))^N D\phi(x+(2s-1)y) ds \\
+ \quad & R^{\frac{-2N}{2N+1}} O(y^{2N+2}/c^2).
\end{aligned}
$$

Thus we have (7.2). Next we show (7.3). We have

$$
\begin{aligned}
\frac{\partial \eta}{\partial R} \quad = \quad & (3) + (4) + (5), \\
(3) \quad = \quad & \frac{2}{2N+1} R^{\frac{-2N}{2N+1}} \int_0^1 K(x,y,x+(2s-1)y) \phi(x+(2s-1)y) ds, \\
(4) \quad = \quad & 2R^{\frac{-2N}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{1}{2N+1} y (K_y + (2s-1)K_\xi)) \times \\
& \phi(x+(2s-1)y) ds, \\
(5) \quad = \quad & 2R^{\frac{-2N}{2N+1}} \int_0^1 K(x,y,x+(2s-1)y)(-x + \frac{y}{2N+1}(2s-1)) D\phi(\ldots) ds.
\end{aligned}
$$

By Proposition 13 we get

$$
(3) = \frac{2^{2N+1}}{2N+1} \int_0^1 (s-s^2)^N \phi(\ldots) ds + O(y^2/c^2).
$$

As for (4) we use Proposition 16 and

$$
K_y \quad + \quad (2s-1)K_\xi = yJ^{N-1}G - (2s-1)(\xi - x)G(x, |\xi - x|, \xi)J^{N-1}1 + (2s-1)J^N G_\xi
$$

$$\begin{aligned} &= 2^{2N+1}N(s-s^2)^N y^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2}(2s-1)(s-s^2)^N y^{2N} + \\ &+ \frac{2^N C_3(x,c)}{N!c^2}(2s-1)(s-s^2)^N y^{2N} + O(y^{2N+1}/c^2) \end{aligned}$$

(See Proposition 19). Then by integration by parts we have

$$(4) = \frac{2^{2N+2}N}{2N+1}\int_0^1 (s-s^2)^N \phi(...)ds + O(y^2/c^2).$$

As (2) we get

$$(5) = 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi(...)ds + O(y^2/c^2).$$

Thus we get (7.3).

Next we show (7.4). We have

$$\begin{aligned} \frac{\partial^2\eta}{\partial M^2} &= (6)+(7)+(8), \\ (6) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 ((K_x+K_\xi)_x + (K_x+K_\xi)_\xi)(x,y,...)\times\phi(...)ds, \\ (7) &= 4R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,...)D\phi(...)ds, \\ (8) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(x,y,...)D^2\phi(...)ds. \end{aligned}$$

By Proposition 18 we have

$$\begin{aligned} ((K_x+K_\xi)_x &+ (K_x+K_\xi)_\xi)(x,y,x+(2s-1)y) = \\ &= \frac{2^N C_2(x,c)}{N!c^2}(s-s^2)^N(2s-1)y^{2N+1} + O(y^{2N+2}/c^2). \end{aligned}$$

Thus by integration by parts we get

$$(6) = O(y^{-2N+1}/c^2).$$

By the same discussion as (1) we see $(7) = O(y^{-2N+1}/c^2)$. By the same discussion as (2) we see

$$(8) = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi(...)ds + O(y^{-2N+1}/c^2).$$

Thus we get (7.4).

Next we show (7.5). We see

$$\frac{\partial^2\eta}{\partial M\partial R} = (9)+(10)+(11)+(12)+(13)+(14),$$

$$\begin{aligned}
(9) &= -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,...)\phi(...)ds,\\
(10) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 (-x(K_x+K_\xi)_x+(K_x+K_\xi)_\xi)+\\
&+ \frac{y}{2N+1}((K_x+K_\xi)_y+(2s-1)(K_x+K_\xi)_\xi))\phi(...)ds,\\
(11) &= 2R^{\frac{4N}{2N+1}}\int_0^1 (K_x+K_\xi)(-x+\frac{y}{2N+1}(2s-1))D\phi ds,\\
(12) &= -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 KD\phi ds,\\
(13) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 (-x(K_x+K_\xi)+\frac{y}{2N+1}(K_y+(2s-1)K_\xi))D\phi ds\\
(14) &= 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds.
\end{aligned}$$

We already know that $(9)=O(y^{-2N+1}/c^2)$. (Recall (1).) Next we look at (10). The first term is $O(y^{-2N+1}/c^2)$. (Recall (6)). By Proposition 16 and 20 we see

$$\begin{aligned}
(K_x+K_\xi)_y &+ (2s-1)(K_x+K_\xi)_\xi =\\
&= \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)(s-s^2)^N+\\
&+ \frac{2^N C_4(x,c)}{N!c^2}y^{2N}(2s-1)(s-s^2)^N+\\
&- \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)^3(s-s^2)^{N-1}+O(y^{2N+1}/c^2).
\end{aligned}$$

By integration by parts we see $(10)=O(y^{-2N+1}/c^2)$. We already know $(11)=O(y^{-2N+1}/c^2)$. Clearly

$$(12)=-\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D\phi ds+O(y^{-2N+1}/c^2).$$

We see

$$(13)=O(y^{-2N+1}/c^2)+\frac{2}{2N+1}R^{\frac{-4N}{2N+1}}\int_0^1 (K_y+(2s-1)K_\xi)D\phi ds.$$

As (4) we have

$$(13)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D\phi ds+O(y^{-2N+1}/c^2).$$

Finally we see

$$(14)=2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+(2s-1)\frac{y}{2N+1})D^2\phi ds+O(y^{-2N+1}/c^2).$$

(Recall (5)). Summing up we get (7.5).

Next we show (7.6).

$$
\begin{aligned}
\frac{\partial^2\eta}{\partial R^2} &= \frac{\partial}{\partial R}(3) + \frac{\partial}{\partial R}(4) + \frac{\partial}{\partial R}(5),\\
\frac{\partial}{\partial R}(3) &= (15) + (16) + (17),\\
(15) &= -\frac{4N}{(2N+1)^2} R^{\frac{-4N-1}{2N+1}} \int_0^1 K\phi ds,\\
(16) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\phi ds,\\
(17) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
\frac{\partial}{\partial R}(4) &= (18) + (19) + (20),\\
(18) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\phi ds,\\
(19) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K''\phi ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K'' &= x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) +\\
&+ \frac{y}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) +\\
&- \frac{xy}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) +\\
&+ \frac{y^2}{2N+1}((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi) +\\
&+ \frac{y}{(2N+1)^2}(K_y + (2s-1)K_\xi),\\
(20) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\\
&\times (-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
\frac{\partial}{\partial R}(5) &= (21) + (22) + (23) + (24),\\
(21) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
(22) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\\
&\times (-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
(23) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(x + \frac{y}{(2N+1)^2}(2s-1))D\phi ds,
\end{aligned}
$$

$$(24) \quad = \quad 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))^2 D^2 \phi ds.$$

First we see

$$(15) \quad = \quad -\frac{-2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$

$$(16) \quad = \quad \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$

$$(17) \quad = \quad \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Thus we have

$$\frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x+\frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Since (18) is similar to (16), we have

$$(18) = -\frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).$$

Next let us look at (19). We already know

$$\begin{aligned} 2R^{\frac{-4N-1}{2N+1}} \int_0^1 x(K_x + K_\xi)\phi ds &= O(y^{-2N+1}/c^2), \\ 2R^{\frac{-4N-1}{2N+1}} \int_0^1 x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)\phi ds &= O(y^{-2N+1}/c^2). \end{aligned}$$

Recalling (10), we see

$$\begin{aligned} \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi)\phi ds &= O(y^{-2N+1}/c^2), \\ \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} xy \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi)\phi ds &= O(y^{-2N+1}/c^2). \end{aligned}$$

When $N = 1$, we have

$$\begin{aligned} (K_y \quad + &\quad (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \\ = &\quad 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3 y - \\ - &\quad \frac{2C_3}{c^2}(2s-1)^3 y + O(y^2/c^2). \end{aligned}$$

When $N \geq 2$, there are bounded functions $F_j(x,c)$ such that

$$\begin{aligned} (K_y \quad + &\quad (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi = \\ = &\quad 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x,c)}{c^2}(2s-1)(s-s^2)^{N-1} y^{2N-1} + \\ + &\quad \frac{F_2(x,c)}{c^2}(2s-1)(s-s^2)^{N-2} y^{2N-1} + \frac{F_3(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-2} y^{2N-1} + \\ + &\quad \frac{F_4(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-1} y^{2N-1} + \frac{F_5(x,c)}{c^2}(2s-1)^5(s-s^2)^{N-2} y^{2N-1} + \\ + &\quad O(y^{2N}/c^2). \end{aligned}$$

Thus we see

$$
\begin{aligned}
& 2R^{\frac{-4N-1}{2N+1}}\frac{y^2}{(2N+1)^2}\int_0^1((K_y+(2s-1)K_\xi)_y+(2s-1)(K_y+(2s-1)K_\xi)_y)\phi ds \\
= & \frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds-\frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds \\
& +O(y^{-2N+1}/c^2).
\end{aligned}
$$

We have

$$
\begin{aligned}
& \frac{2}{(2N+1)^2}R^{\frac{-4N-1}{2N+1}}y\int_0^1(K_y+(2s-1)K_\xi)\phi ds \\
= & \frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Therefore

$$
(19)=\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).
$$

We see

$$
(20)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).
$$

Therefore

$$
\frac{\partial}{\partial R}(4)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).
$$

Next we see

$$
\begin{aligned}
(21) &= -\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2), \\
(22) &= \frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2), \\
(23) &= 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+O(y^{-2N+1}/c^2), \\
(24) &= 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+}(2s-1))^2D^2\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\frac{\partial}{\partial R}(5) &= 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+ \\
&+ 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds+O(y^{-2N+1}/c^2).
\end{aligned}
$$

Summing up, we have

$$
\frac{\partial^2\eta}{\partial R^2} = 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds
$$

$$
\begin{aligned}
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds \\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds \\
&= \quad \frac{2^{2N+2}(N+1)}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^N(2s-1)yD\phi ds + \\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds \\
&= \quad \frac{2^{2N+3}}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^{N+1}y^2D^2\phi ds + \\
&+ \quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds.
\end{aligned}
$$

Thus we get (7.6). QED.

Let us recall the standard entropy η^*. This is generated by

$$\phi^*(x) = A'c^2(\frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}}),$$

where

$$A' = (2N+1)^{-2N}((2N+1)/(2N+3)A)^{\frac{2N+1}{2}}(2N-1)!!/2^{N+1}N!.$$

We note that

$$D^2\phi^*(x) = A'(1+\frac{u^2/c^2}{1-u^2/c^2})(2-\sqrt{1-u^2/c^2}) \geq A'.$$

We are going to show that the Hessian $D_U^2\eta^*$ dominates any $D_U^2\eta$.

Proposition 22 *For each ϕ fixed in C^3 we have on each compact subset of $\{\rho \geq 0\}$*

$$|(\xi|D_U^2\eta.\xi)| \leq C(\xi|D_U^2\eta^*.\xi),$$

provided that c is sufficiently large.

By the assumption we have

$$
\begin{aligned}
R &= y^{2N+1} = K\rho(1+[\rho^{\frac{2}{2N+1}}/c^2]_1), \\
\frac{dR}{d\rho} &= K + [\rho^{\frac{2}{2N+1}}/c^2]_1, \\
\frac{d^2R}{d\rho^2} &= \frac{\rho^{\frac{1-2N}{2N+1}}}{c^2}[\rho^{\frac{2}{2N+1}}/c^2]_0,
\end{aligned}
$$

where $K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}}$. Using these, we have

$$\frac{\partial R}{\partial E} = \frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4}$$

$$
\begin{aligned}
&= K(1+u^2/c^2)+O(y^2/c^2),\\
\frac{\partial R}{\partial F} &= -\frac{dR}{d\rho}\frac{2u/c^2}{1-P'u^2/c^4}\\
&= -K\frac{2u}{c^2}+O(y^2/c^2),\\
\frac{\partial M}{\partial E} &= -\frac{R}{\rho+P/c^2}\frac{1+P'/c^2}{1-P'u^2/c^4}u+x\frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4}\\
&= K(-u+x(1+u^2/c^2))+O(y^2/c^2),\\
\frac{\partial M}{\partial F} &= \frac{R}{\rho+P/c^2}\frac{1+P'u^2/c^4}{1-P'u^2/c^4}-\frac{dR}{d\rho}2xu/c^2\frac{1}{1-P'u^2/c^4}\\
&= K(1-2xu/c^2)+O(y^2/c^2).
\end{aligned}
\tag{7.7}
$$

Differentiating once more, we see

$$
\begin{aligned}
\frac{\partial^2 R}{\partial E^2} &= -\frac{K^2}{y^{2N+1}}2u^2(1-u^2/c^2)/c^2+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial E^2} &= \frac{K^2}{y^{2N+1}}u(-2u^2/c^2-2ux(1-u^2/c^2)/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 R}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}\frac{2u}{c^2}(1-u^2/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}(2u^2/c^2+2xu(1-u^2/c^2)/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 R}{\partial F^2} &= -\frac{2}{c^2}\frac{K^2}{y^{2N+1}}(1-u^2/c^2)+O(y^{-2N+1}/c^2),\\
\frac{\partial^2 M}{\partial F^2} &= -\frac{K^2}{y^{2N+1}}2(u+x(1-u^2/c^2))/c^2+O(y^{-2N+1}/c^2).
\end{aligned}
\tag{7.8}
$$

The chain rule gives

$$
\begin{aligned}
\frac{\partial^2\eta}{\partial E^2} &= (\frac{\partial R}{\partial E})^2\frac{\partial^2\eta}{\partial R^2}+2\frac{\partial R}{\partial E}\frac{\partial M}{\partial E}\frac{\partial^2\eta}{\partial R\partial M}\\
&+ (\frac{\partial M}{\partial E})^2\frac{\partial^2\eta}{\partial M^2}+\frac{\partial^2 R}{\partial E^2}\frac{\partial\eta}{\partial R}+\frac{\partial^2 M}{\partial E^2}\frac{\partial\eta}{\partial M},
\end{aligned}
\tag{7.9}
$$

and so on. Inserting (7.7)nd (7.8) into (7.9), and using Proposition 21, we have

$$
\begin{aligned}
(\xi|D_U^2\eta.\xi) &= \frac{2^{2N+1}K^2}{y^{2N+1}}\int_0^1(s-s^2)^N Z[\xi]D^2\phi ds+\\
&- \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial R}+\\
&- \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial M}+\\
&+ O(y^{-2N+1}/c^2),
\end{aligned}
$$

where

$$
Z[\xi] = Z_{00}\xi_0^2+2Z_{01}\xi_0\xi_1+Z_{11}\xi_1^2,
$$

$$
\begin{aligned}
Z_{00} &= (1+u^2/c^2)^2((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+ \\
&+ 2(1+u^2/c^2)(-u+x(1+u^2/c^2))(-x+\frac{y}{2N+1}(2s-1)) \\
&+ (-u+x(1+u^2/c^2))^2, \\
Z_{01} &= -2(1+u^2/c^2)u/c^2((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+ \\
&+ (1+3u^2/c^2-4x(1+u^2/c^2)u/c^2)(-x+\frac{y}{2N+1}(2s-1))+ \\
&+ (-u+x(1+u^2/c^2))(1-2xu/c^2), \\
Z_{11} &= \frac{4u^2}{c^4}((-x+\frac{y}{2N+1}(2s-1))^2+\frac{4}{(2N+1)^2}s(1-s)y^2)+ \\
&- \frac{4u}{c^2}(1-2xu/c^2)(-x+\frac{y}{2N+1}(2s-1))+ \\
&+ (1-2xu/c^2)^2.
\end{aligned}
$$

It can be shown that

$$Z[\xi] \geq \kappa s(1-s)y^2,$$

where κ is a positive constant depending on the compact subset of $\{\rho \geq 0\}$. In fact we see

$$Z_{00}Z_{11} - Z_{01}^2 = (1-u^2/c^2)\frac{4}{(2N+1)^2}s(1-s)y^2.$$

On the other hand, we can estimate

$$
\begin{aligned}
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)\frac{\partial\eta}{\partial R}| &\leq \frac{\epsilon}{y^{2N+1}}, \\
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))\frac{\partial\eta}{\partial M}| &\leq \frac{\epsilon}{y^{2N+1}},
\end{aligned}
$$

where $\epsilon = K'/c^2$. Let us introduce the parameters

$$\zeta_0 = \xi_0, \qquad \zeta_1 = \xi_1 - u\xi_0.$$

Then we have

$$Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2,$$

and

$$
\begin{aligned}
Q_{00} &= Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x,s)y^2, \\
Q_{01} &= Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x,s)y^2, \\
Q_{11} &= Z_{11} = 1 + O(1/c^2) > 0.
\end{aligned}
$$

Therefore if $|D^2\phi| \leq C$, we see

$$|(\xi|D_U^2\eta\xi)| \leq \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds$$

$$
\begin{aligned}
&+ \quad \frac{12\epsilon}{y^{2N+1}}\int_0^1 (s-s^2)^N\zeta^2 ds + O(y^{-2N+1}/c^2) \\
&\le \quad \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1 (s-s^2)^N(Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00}\zeta_0^2)ds \\
&+ \quad O(y^{-2N+1}/c^2).
\end{aligned}
$$

But since $Q_{00}^{(0)} = Q_{01}^{(0)} = 0$, $\int_0^1 (s-s^2)^N(2s-1)ds = 0$, we see

$$
\int_0^1 (s-s^2)^N(-2\epsilon' Q_{01}\zeta_0\zeta_1 - \epsilon' Q_{00}\zeta_0^2)ds = O(y^{-2N+1}/c^2).
$$

Therefore we get

$$
|(\xi|D_U^2\eta\xi)| \le \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).
$$

Similarly, if $D^2\phi^* \ge \mu$, we have

$$
(\xi|D_U^2\eta^*\xi) \ge \frac{2^{2N+1}K^2\mu(1-\epsilon'')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).
$$

Thus we get

$$
|(\xi|D_U^2\eta\xi)| \le \frac{C(1+\epsilon')}{\mu(1-\epsilon'')}(\xi|D_U^2\eta^*\xi) + O(y^{-2N+1}/c^2).
$$

But we know

$$
(\xi|D_U^2\eta^*\xi) \ge \kappa|\xi|^2 y^{-2N+1}.
$$

Hence if c is sufficiently large we get the required estimate. QED.

As for the first derivatives, the following conclusion is now clear.

Proposition 23 *On each compact subset of* $\{\rho \ge 0\}$, *we have*

$$
|\frac{\partial\eta}{\partial E}| + |\frac{\partial\eta}{\partial F}| \le C.
$$

8 Usefull entropies

Let us consider an entropy η generated by ϕ, that is,

$$
\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi. \tag{8.1}
$$

The corresponding entropy flux q is given by integrating the differential equations

$$
\frac{\partial q}{\partial w} = \lambda_2\frac{\partial\eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1\frac{\partial\eta}{\partial z}.
$$

We can solve these equations as

$$\begin{aligned} q &= \lambda_2\eta - \int_z^w \frac{\partial\lambda_2}{\partial w}\eta dw \\ &= \lambda_1\eta + \int_z^w \frac{\partial\lambda_1}{\partial z}\eta dz. \end{aligned}$$

Thus we get the formula

$$q(x,y) = \int_{x-y}^{x+y} L(x,y,\xi)\phi(\xi)d\xi, \tag{8.2}$$

where

$$\begin{aligned} L(x,y,\xi) &= \lambda_1 K(x,y,\xi) + L_1(x,y,\xi) \\ &= \lambda_2 K(x,y,\xi) + L_2(x,y,\xi), \\ L_1(x,y,\xi) &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y,)K(x+y-Y,Y,\xi)dY, \\ L_2(x,y,\xi) &= -2\int_{(-x+y+\xi)/2}^{y} \mu_2(x-y+Y,Y)K(x-y+Y,Y,\xi)dY, \\ \mu_1(x,y) &= \frac{\partial\lambda_1}{\partial z} \\ &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N}{2N+1}+O(1/c^2), \\ \mu_2(x,y) &= \frac{\partial\lambda_2}{\partial w} \\ &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)}(1-\frac{P'}{c^2}+\frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N}{2N+1}+O(1/c^2). \end{aligned}$$

In this section we will construct various kinds of usefull entropies.

1) Let us put

$$\begin{aligned} \eta_k^1(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{k\xi}d\xi, \\ \eta_k^2(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{-k\xi}d\xi. \end{aligned}$$

Proposition 24 *If $1/c^2$ is sufficiently small, we have*

$$\eta_k^1 > 0, \qquad \eta_k^2 > 0 \qquad \textit{for } y > 0, \tag{8.3}$$

$$\begin{aligned} \eta_k^1 &= 2^N N! y^N (1+O(y/c^2))e^{k(x+y)}(1+O(1/k)), \\ \eta_k^2 &= 2^N N! y^N (1+O(y/c^2))e^{-k(x-y)}(1+O(1/k)) \end{aligned} \tag{8.4}$$

uniformly on each compact subset of $\{y > 0\}$. *Moreover*

$$\begin{aligned} q_k^1 &= \eta_k^1(\lambda_2 + O(1/k)), \\ q_k^2 &= \eta_k^2(\lambda_1 + O(1/k)) \end{aligned} \tag{8.5}$$

uniformly on each compact subset of $\{y \geq 0\}$ *and*

$$\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} (\frac{1}{2N+1} + O(1/c^2)) e^{2ky} (y + O(1/k))^3. \tag{8.6}$$

Proof. Since $K = (1 + O(y/c^2))(y^2 - (x-\xi)^2)^N$, we see

$$\begin{aligned} \eta_k^1 &= (1 + O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N k^{N+1} e^{k\xi} d\xi \\ &= (1 + O(y/c^2)) 2^{2N+1} y^N e^{kx} f(ky) \end{aligned}$$

where

$$\begin{aligned} f(r) &= r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds \\ &= e^r \int_0^r (\sigma(1 - \frac{\sigma}{r}))^N e^{-2\sigma} d\sigma. \end{aligned}$$

It is easy to see

$$e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)$$

This implies (8.4). We note

$$\begin{aligned} \eta^1 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k)) \\ \eta^2 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k)) \end{aligned}$$

uniformly on $\{y \geq 0\}$. Let us consider the flux. We have

$$\begin{aligned} L_2(x,y,\xi) &= -2 \int_{(-x+y+\xi)/2}^{y} \mu_2(x-y+Y,Y) K(x-y+Y,Y,\xi) dY \\ &= -2(\frac{N}{2N+1} + O(1/c^2)) \int_{(-x+y+\xi)/2}^{y} (Y^2 - (x-y+Y-\xi)^2)^N dY \\ &= -(\frac{N}{(2N+1)(N+1)} + O(1/c^2))(y-x+\xi)^N (y+x-\xi)^{N+1}, \\ q^1 - \lambda_2 \eta^1 &= -(\frac{N}{(2N+1)(N+1)} + O(1/c^2)) \int_{x-y}^{x+y} (y-x+\xi)^N (y+x-\xi)^{N+1} k^{N+1} e^{k\xi} d\xi. \end{aligned}$$

But

$$\begin{aligned} 0 &\leq \int_{x-y}^{x+y} (y-x+\xi)^N (y+x-\xi)^{N+1} k^{N+1} e^{k\xi} d\xi \\ &= (N+1) k^N \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N e^{k\xi} d\xi \end{aligned}$$

$$\begin{aligned}
&- \quad Nk^N \int_{x-y}^{x+y} (y-x+\xi)^{N-1}(y+x-\xi)^{N+1}e^{k\xi}d\xi \\
&\leq \quad (N+1)\frac{1}{k}\int_{x-y}^{x+y}(y^2-(x-\xi)^2)^N k^{N+1}e^{k\xi}d\xi.
\end{aligned}$$

Thus

$$q^1 - \lambda_2\eta^1 = O(1/k)\eta^1.$$

Since

$$\lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1-u^2/c^2)}{1-P'u^2/c^4} = (\frac{1}{2N+1} + O(1/c^2))y,$$

we have

$$\eta^2 q^1 - \eta^1 q^2 = \eta^1\eta^2((\frac{1}{2N+1} + O(1/c^2))y + O(1/k)).$$

This implies (8.6). QED.

2) Let ψ be a function in $C_0^\infty(-1,1)$ such that $\psi \geq 0, \int \psi = 1$. We put

$$\begin{aligned}
\phi_n^3(x) &= \psi_n(x) = n\psi(n(x-a)), \\
\phi_n^4(x) &= -D\psi_n(x), \\
\eta_n^3(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^3(\xi)d\xi, \\
\eta_n^4(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^4(\xi)d\xi. \\
\eta^3(x,y) &= K(x,y,a)X, \\
\eta^4(x,y) &= K_\xi(x,y,a)X, \\
q^3(x,y) &= L(x,y,a)X, \\
q^4(x,y) &= L_\xi(x,y,a)X, \\
X &= 1 \qquad (x-y<a<x+y) \\
&= \frac{1}{2} \qquad (|x-a|=y) \\
&= 0 \qquad (|x-a|>y).
\end{aligned}$$

Proposition 25 *As $n \to \infty$, we have*

$$\eta_n^3 \to \eta^3, \qquad q_n^3 \to q^3, \qquad \eta_n^4 \to \eta^4, \qquad q_n^4 \to q^4.$$

Moreover

$$\begin{aligned}
|\eta_n^3| &\leq My^{2N}, \qquad |q_n^3| \leq My^{2N}(|x|+y), &(8.7)\\
|\eta_n^4| &\leq My^{2N-1}, \qquad |q_n^4| \leq My^{2N-1}(|x|+y), &(8.8)\\
\eta^3q^4 - \eta^4q^3 &= \frac{N}{(2N+1)(N+1)}(1+O(1/c^2))(y^2-(x-a)^2)^{2N} &(8.9)
\end{aligned}$$

Proof. We note

$$\begin{aligned}
K_\xi &= -(\xi - x)G(x,|\xi - x|,\xi)\frac{1}{2^{N-1}(N-1)!}(y^2 - (x-\xi)^2)^{N-1} + J^N G_\xi \\
&= (2N(x-\xi) + O(1/c^2)(\xi - x)^2)(y^2 - (x-\xi)^2)^{N-1} + O(1/c^2)(y^2 - (x-\xi)^2)^N, \\
L_{1,\xi} &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y)K_\xi(x+y-Y,Y,\xi)dY.
\end{aligned}$$

The estimates (8.7), (8.8) can be seen easily. Let us consider

$$\eta^3 q^4 - \eta^4 q^3 = (KL_\xi - LK_\xi)(x,y,a).$$

Suppose $x - a \geq 0$. Then

$$\begin{aligned}
\frac{1}{2}(KL_\xi - LK_\xi) &= K\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY - \\
&- K_\xi \int_{(x+y-a)/2}^{y} \mu_1 K(x+y-Y,Y,a)dY.
\end{aligned}$$

We note

$$0 \leq \frac{x+y-a}{2} \leq x - y + Y - a \leq x - a \leq y.$$

Hence we have

$$\begin{aligned}
&\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY \\
&= (\frac{N}{2N+1} + O(1/c^2))2N\int_{(x+y-a)/2}^{y}(x+y-Y-a)(Y^2 - (x+y-Y-a)^2)^{N-1}dY + \\
&+ O(1/c^2)\int_{(x+y-a)/2}^{y}(Y^2 - (x+y-Y-a)^2)^N dY \\
&= (\frac{N^2}{2(2N+1)} + O(1/c^2))(x+y-a)^{N-1}(-x+y+a)^N\frac{1}{N(N+1)}(y + (2N+1)(x-a)) \\
&+ O(1/c^2)(y^2 - (x-a)^2)^N.
\end{aligned}$$

Thus

$$\begin{aligned}
&K\int_{(x+y-a)/2}^{y} \mu_1 K_\xi dY \\
&= (\frac{N}{2(2N+1)(N+1)} + O(1/c^2))(y^2 - (x-a)^2)^{2N-1}(-x+y+a)(y + (2N+1)(x-a)) \\
&+ O(1/c^2)(y^2 - (x-a)^2)^{2N}.
\end{aligned}$$

Also we have

$$\begin{aligned}
&K_\xi \int_{(x+y-a)/2}^{y} \mu_1 K dY \\
&= (\frac{N^2}{(2N+1)(N+1)} + O(1/c^2))(x-a)(-x+y+a)(y^2 - (x-a)^2)^{2N-1} \\
&+ O(1/c^2)(-x+y+a)(y^2 - (x-a)^2)^{2N}.
\end{aligned}$$

Hence

$$\frac{1}{2}(KL_\xi - LK_\xi) = (\frac{N}{2(2N+1)(N+1)} + O(1/c^2))(y^2 - (x-a)^2)^{2N}.$$

Here we have used

$$\begin{aligned} 0 &\leq (x-a)(y-(x-a)) \leq y^2 - (x-a)^2, \\ 0 &\leq (y-x+a)(y+(2N+1)(x-a)) \\ &\leq (2N+1)(y^2-(x-a)^2) \end{aligned}$$

provided that $0 \leq x - a \leq y$. When $x - a \leq 0$, we can discuss in a similar manner by using L_2. QED.

3) Let Φ be a function in $C_0^\infty(-1,1)$ such that $\int \Phi = 0$ and the support $supp\Phi$ is $[-1+\alpha, 1+\alpha]$, where α is a small positive number. We put

$$\begin{aligned} \psi_n(x) &= n\Phi(n(x-a)), \\ \eta_n^5(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi, \\ q_n^5(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi; \\ \hat{\Phi}(x) &= \frac{d}{dx}(x\int_{-1}^{x}\Phi), \\ \hat{\psi}_n(x) &= n\hat{\Phi}(n(x-a)), \\ \eta_n^6(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi d\xi, \\ q_n^6(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi; \\ B_n^3 &= \eta^3 q_n^5 - \eta_n^5 q^3, \\ B_n^4 &= \eta^4 q_n^5 - \eta_n^5 q^4, \\ B_n &= \eta_n^5 q_n^6 - \eta_n^6 q_n^5. \end{aligned}$$

Let us divide the domain $\Sigma = \{-B \leq x - y \leq x + y \leq B\}$ into the following 5 parts.

$$\begin{aligned} S_0 &= \{-\frac{1}{n} < x+y-a \leq \frac{1}{n}, -\frac{1}{n} \leq x-y-a < \frac{1}{n}\} \cap \Sigma, \\ S_1 &= \{\frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n}\} \cap \Sigma, \\ S_L &= \{-\frac{1}{n} < x+y-a \leq \frac{1}{n}, x-y-a < -\frac{1}{n}\} \cap \Sigma, \\ S_R &= \{\frac{1}{n} < x+y-a, -\frac{1}{n} \leq x-y-a < \frac{1}{n}\} \cap \Sigma, \\ S &= \Sigma - (S_0 \cup S_1 \cup S_1 \cup S_L \cup S_R). \end{aligned}$$

Proposition 26 *We have*

$$|B_n^3| \leq M/n, \qquad |B_n^4| \leq M \tag{8.10}$$

on Σ, and

$$|B_n| \leq M/n \tag{8.11}$$

on $S_0 \cup S_1 \cup S$. Moreover, on S_L, we have

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3, \tag{8.12}$$

where

$$\begin{aligned} A_1 &= (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2, \\ |A_2| &\leq M(|\int_{-1}^{n(x+y-a)} \Phi| + |\Phi(n(x+y-a))|), \\ |A_3| &\leq \frac{M}{n}. \end{aligned}$$

On S_R, we have

$$\begin{aligned} B_n &= ny^{2N}C_1 + y^N C_2 + C_3, \\ C_1 &= (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x-y-a)} \Phi)^2, \\ |C_2| &\leq M(|\int_{-1}^{n(x-y-a)} \Phi| + |\Phi(n(x-y-a))|), \\ |C_3| &\leq \frac{M}{n}. \end{aligned}$$

Proof. For the simplicity, we write $\eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6$. It is easy to see inductively that, for $G_j = J^j G = K_{N-j}$, we have

$$\partial_\xi^p G_j = J\partial_\xi^p G_{j-1}$$

for $j \geq p+1$ and

$$\partial_\xi^p G_p = (-1)^p(\xi - x)^p G(x, |\xi - x|, \xi) + J\partial_\xi^p G_{p-1}.$$

Therefore

$$\partial_\xi^p K = \partial_\xi^p G_N(x, y, \xi) = 0$$

for $p \leq N-1$ and $y = |x - \xi|$. Thus by integration by parts we have

$$\begin{aligned} \eta_n &= (-1)^N \partial_\xi^N K(x, y, x+y)\psi_n(x+y) + \\ &- (-1)^N \partial_\xi^N K(x, y, x-y)\psi_n(x-y) + \\ &+ F_n^1(x, y), \\ F_n^1(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi)\psi_n(\xi)d\xi. \end{aligned}$$

We see

$$\partial_\xi^p L_2(x,y,\xi) = -2\int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_\xi^p K(x-y+Y,Y,\xi)dY$$

for $p \le N-1$. Therefore

$$\partial_\xi^p L_2(x,y,x+y) = \partial_\xi^p L_2(x,y,x-y) = 0$$

for $p \le N-1$. Moreover we see

$$\partial_\xi^N L_2(x,y,x+y) = 0.$$

Therefore by integration by parts we have

$$\begin{aligned}
\sigma_n(x,y) &= q_n(x,y) - \lambda_2\eta_n(x,y) \\
&= -(-1)^N \partial_\xi^N L_2(x,y,x-y)\psi_n(x-y) + \\
&+ F_n^2(x,y), \\
F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x,y,\xi)\psi_n(\xi)d\xi.
\end{aligned}$$

Similarly

$$\begin{aligned}
\bar\sigma_n(x,y) &= q_n(x,y) - \lambda_1\eta_n(x,y) = \\
&= (-1)^N \partial_\xi^N L_1(x,y,x+y)\psi_n(x+y) + \\
&+ \bar F_n^2(x,y), \\
\bar F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_1(x,y,\xi)\psi_n(\xi)d\xi.
\end{aligned}$$

We note

$$\partial_\xi^N K(x,y,\xi) = (-1)^N(\xi-x)^N G(x,|x-\xi|,\xi) + J\partial_\xi^N G_{n-1}.$$

It is easy to see inductively that

$$\begin{aligned}
\partial_\xi^{p+1} G_p(x,y,\xi) &= (-1)^p \frac{p(p+1)}{2}(\xi-x)^{p-1}G(x,|x-\xi|,\xi) + \\
&+ (\xi-x)^p H_p(x,\xi) + J\partial_\xi^p G_{p-1},
\end{aligned}$$

where $H_p = O(1/c^2)$. Therefore

$$\begin{aligned}
\partial_\xi^{N+1} K(x,y,\xi) &= (-1)^N \frac{N(N+1)}{2}(\xi-x)^{N-1}G(x,|\xi-x|,\xi) + \\
&+ (\xi-x)^N H_N(x,\xi) + J\partial_\xi^N G_{N-1}.
\end{aligned}$$

1) Suppose $(x,y) \in S$. Then it is clear that η^3, η^4, q^3, q^4, η_n, q_n, $\hat\eta_n, \hat q_n$, B_n^3, B_n^4, B_n all vanish.

2) Suppose $(x,y)\in S_0$. Then we see

$$\begin{aligned}
\eta^3 &= K(x,y,a)\\
&= O((y^2-(x-a)^2)^N)\\
&= O(n^{-2N}),\\
\eta^4 &= K_\xi(x,y,a)\\
&= O(|x-a|(y^2-(x-a)^2)^{N-1})+O((y^2-(x-a)^2)^N)\\
&= O(n^{-2N+1}),\\
\sigma^3 &= L_2(x,y,a)\\
&= -2\int_{(-x+y+a)/2}^{y}\mu_2K(x-y+Y,Y,a)dY\\
&= O(n^{-2N-1}),\\
\sigma^4 &= L_{2,\xi}(x,y,a)\\
&= -2\int_{(-x+y+a)/2}^{y}\mu_2K_\xi(x-y+Y,Y,a)dY\\
&= O(n^{-2N}).
\end{aligned}$$

Since $y=O(1/n)$ and $\psi_n=O(n)$, we see

$$\begin{aligned}
&(-1)^N\partial_\xi^N K(x,y,x+y)\psi_n(x+y)+\\
-\;&(-1)^N\partial_\xi^N K(x,y,x-y)\psi_n(x-y)=\\
=\;&O(n^{-N+1}).
\end{aligned}$$

Since $F_n^1=O(1)$, we have $\eta_n=O(1)$. We see

$$\partial_\xi^N L_2(x,y,x-y)=-2\int_0^y \mu_2\partial_\xi^N K(x-y+Y,Y,x-y)dY=O(n^{-N-1}).$$

Therefore

$$-(-1)^N\partial_\xi^N L_2(x,y,x-y)\psi_n(x-y)=O(n^{-N}).$$

Since

$$\begin{aligned}
\partial_\xi^{N+1}L_2(x,y,\xi) &= \mu_2\partial_\xi^N K((x-y+\xi)/2,(-x+y+\xi)/2,\xi)+\\
&- 2\int_{(-x+y+\xi)/2}^{y}\partial_\xi^{N+1}K(x-y+Y,Y,\xi)dY\\
&= O((-x+y+\xi)^N)+O(x+y-\xi),
\end{aligned}$$

we see

$$\begin{aligned}
F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}L_2(x,y,\xi)\psi_n(\xi)d\xi\\
&= O(n^{-1}).
\end{aligned}$$

Hence $\sigma_n = O(n^{-1})$. Therefore

$$\begin{aligned} B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-2N-1}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{-2N}), \\ B_n &= \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_n = O(n^{-1}). \end{aligned}$$

3) Suppose $(x,y) \in S_1$, where $x+y > a+\frac{1}{n}$ and $x-y < a-\frac{1}{n}$. Then $\psi_n(x+y) = \psi_n(x-y) = \hat{\psi}_n(x+y) = \hat{\psi}_n(x-y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

$$\begin{aligned} F_n^1(x,y) = (-1)^{N+1} &\int_{x-y}^{x+y} \partial_\xi^{N+1} K(x,y,\xi)\psi_n(\xi)d\xi \\ &= (-1)^{N+1}\int_{-1}^{1}(\partial_\xi^{N+1}K(x,y,a+\frac{s}{n}) - \partial_\xi^{N+1}K(x,y,a))\Phi(s)ds \\ &= O(1/n) \end{aligned}$$

since $\int \Phi = 0$ and $\partial_\xi^{N+1}K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

$$\begin{aligned} B_n^3 &= \eta^3 F_n^2 - F_n^1\sigma^3 = O(1/n), \\ B_n^4 &= \eta^4 F_n^2 - F_n^1\sigma^4 = O(1/n), \\ B_n &= F_n^1\hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2). \end{aligned}$$

4) Suppose $(x,y) \in S_L$, where $|x+y-a| \leq 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(x-y-a) < -1$, we have $\psi_n(x-y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

$$\begin{aligned} B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-N}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{1-N}). \end{aligned}$$

Let us estimate $B_n = \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_N$. Since

$$\begin{aligned} \partial_\xi^{N+1}K &= (-1)^N\frac{N(N+1)}{2}(\xi-x)^{N-1}G(x,|x-\xi|,\xi) + \\ &+ (\xi-x)^N H_N(x,\xi) + J\partial_\xi^N G_{N-1}, \end{aligned}$$

we have

$$\begin{aligned} F_n^1 &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}K(x,y,\xi)\psi_n(\xi)d\xi = \\ &= (-1)^{N+1}((-1)^N\frac{N(N+1)}{2}2^N N!(a-x)^{N-1} + F'(x,a))\int_{-1}^{n(x+y-a)}\Phi + \\ &+ O(1/n) = \\ &= -\frac{N(N+1)}{2}2^N N! y^{N-1}(1+F''(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + \\ &+ O(1/n), \end{aligned}$$

where $F' = O(1/c^2)|x-a|^N, F'' = O(1/c^2)$. On the other hand

$$\partial_\xi^N K(x,y,x+y) = (-1)^N y^N G(x,y,x+y).$$

Hence

$$\begin{aligned}\eta_n &= ny^N G(x,y,x+y)\Phi(n(x+y-a)) + \\ &- \frac{N(N+1)}{2}2^N N! y^{N-1}(1+F''(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + \\ &+ O(1/n).\end{aligned}$$

Since

$$\begin{aligned}\partial_\xi^{N+1} L_2(x,y,\xi) &= \mu_2 \partial_\xi^N K((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\ &- 2\int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_\xi^{N+1} K(x-y+Y, Y, \xi)dY = \\ &= (\frac{N}{2N+1} + O(1/c^2))(-1)^N(\frac{-x+y+\xi}{2})^N \times \\ &\times G((x+y+\xi)/2, (-x+y+\xi)/2, \xi) + \\ &+ O(x+y-\xi),\end{aligned}$$

we see

$$\begin{aligned}\sigma_n &= F_n^2 = \\ &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x,y,\xi)\psi_n(\xi)d\xi = \\ &= -\frac{N}{2N+1}2^N N! y^N(1+L'(x,y,a))\int_{-1}^{n(x+y-a)}\Phi \\ &+ O(1/n),\end{aligned}$$

where $L' = O(1/c^2)$. Here we have used

$$(\frac{-x+y+a}{2})^N = (y - \frac{x+y-a}{2})^N = y^N + O(1/n).$$

Similar estimates hold for $\hat{\eta}_n, \hat{\sigma}_n$. Thus

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3,$$

where

$$\begin{aligned}A_1 &= -G\frac{N}{2N+1}2^N N!(1+L')\Phi(\beta)\int_{-1}^{\beta}\hat{\Phi} + \\ &+ G\frac{N}{2N+1}2^N N!(1+L')\hat{\Phi}(\beta)\int_{-1}^{\beta}\Phi = \\ &= \frac{N}{2N+1}2^N N! G(1+L')(\int_{-1}^{\beta}\Phi)^2, \\ \beta &= n(x+y-a).\end{aligned}$$

The estimates on S_R can be obtained in a similar manner considering $\bar{\sigma}^3, \bar{\sigma}^4, \bar{\sigma}_n$. QED.

If we put

$$\begin{aligned}\hat{B}_n^3 &= \eta^3\eta_n^6 - \eta_n^6 q^3,\\ \hat{B}_n^4 &= \eta^4 q_n^6 - \eta_n^6 q^4,\end{aligned}$$

then the same estimates hold.

9 Compactness of $\eta_t + q_x$

Let us consider an entropy η generated by ϕ through the generalized Darboux formula and its flux q. In this section we will prove

Lemma 1 *Let U^Δ be the approximate solutions constructed in Section 4. Then $\eta(U^\Delta)_t + q(U^\Delta)_x$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$, Ω being a bounded open subset of $\{t \geq 0\}$.*

Proof. Let Φ be a test function and we consider

$$\begin{aligned} J &= \int\int (\eta(U^\Delta)\Phi_t + q(U^\Delta)\Phi_x)dxdt\\ &= N + L + \Sigma,\\ N &= -\int \eta(U^\Delta(+0,x)\Phi(0,x)dx,\\ L &= \sum_n \int [\eta(U^\Delta(t,x)]_{t=n\Delta t+0}^{t=n\Delta t-0}\Phi(n\Delta t,x)dx,\\ \Sigma &= \int \sum_{shock}(\sigma[\eta]-[q])\Phi dt.\end{aligned}$$

Since U^Δ is bounded, we see

$$|N| \leq M||\Phi||_C.$$

Let us look at L. We see

$$\begin{aligned} L &= L_1 + L_2,\\ L_1 &= \sum_{j,n}\Phi(n\Delta t,(2j+1)\Delta x)\int_{2j\Delta x}^{(2j+2)\Delta x}[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0}dx,\\ L_2 &= \sum_{j,n}\int_{2j\Delta x}^{(2j+2)\Delta x}(\Phi(n\Delta t,x)-\Phi(n\Delta t,(2j+1)\Delta x)\times\\ &\times\ [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0}dx.\end{aligned}$$

We note

$$\begin{aligned}
[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} &= D_U\eta(U^\Delta(n\Delta t+0,x))[U^\Delta] \\
&+ \int_0^1 (1-\theta)([U^\Delta]|D_U^2(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta])d\theta.
\end{aligned}$$

and

$$\int_{2j\Delta x}^{(2j+2)\Delta x} [U^\Delta]dx = 0$$

by the scheme. Therefore

$$|L_1| \le M||\Phi||_C \sum_{j,n} \int \int_0^1 (1-\theta)|F(\theta,\eta)|d\theta dx,$$

where

$$F(\theta,\eta) = ([U^\Delta]|D_U^2\eta(U^\Delta(n\Delta t+0)+\theta[U^\Delta]).[U^\Delta]).$$

By Proposition 22 we know $|F(\theta,\eta)| \le MF(\theta,\eta^*)$. But in the proof of Proposition 7 we know

$$\sum_{j,n} \int \int_0^1 (1-\theta)F(\theta,\eta^*)d\theta dx \le C.$$

Thus we know

$$|L_1| \le M||\Phi||_C.$$

In the proof of Proposition 7 we know

$$\sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} |[U^\Delta]|^2 dx \le C.$$

Therefore

$$\begin{aligned}
|L_2| &\le 2^\alpha ||\Phi||_{C^\alpha} \sum_n \int (\Delta x)^\alpha |[\eta(U^\Delta)]|dx \\
&\le 2^{\alpha-1}||\Phi||_{C^\alpha} \sum_n \int ((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}}|[\eta(U^\Delta)]|^2)dx \\
&\le M||\Phi||_{C^\alpha}((\Delta x)^{\alpha-\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} \sum \int |[U^\Delta]|^2 dx \\
&\le M'(\Delta x)^{\alpha-\frac{1}{2}}||\Phi||_{C^\alpha},
\end{aligned}$$

where we use the boundedness of $D_U\eta$ and $n = O(1/(\Delta x))$. Next we look at Σ. Along the shock we have

$$\begin{aligned}
\sigma[\eta(U)] &- [q(U)] \\
&= \int_{\rho_L}^{\rho_R} (-\frac{d\sigma}{d\rho} \int_0^1 \theta(U-U_L|D_U^2\eta(U_L+\theta(U-U_L))(U-U_L))d\theta)d\rho.
\end{aligned}$$

This implies

$$|\sigma[\eta] - [q]| \leq M(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{shock} (\sigma[\eta^*] - [q^*])dt \leq C$$

in the proof of Proposition 7. Therefore

$$|\Sigma| \leq M||\Phi||_C.$$

Summing up , we know the compactness. QED.

10 Convergence of approximate solutions

We consider the approximate solutions U^Δ constructed in Section 4. Since U^Δ is bounded, there is a sequence U^{Δ_n} and a family of Young measures $\nu_{t,x}$ such that $supp\nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function f

$$f(U^{\Delta_n}(t,x)) \rightarrow \bar{f} = < \nu_{t,x}, f >$$

in L^∞ weak star topology. By Lemma 1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \rightarrow < \nu, q > < \nu, q' > - < \nu, \eta' > < \nu, q >$$

in L^∞ weak star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

Lemma 2 *For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity*

$$< \nu, \eta q' - \eta' q > = < \nu, \eta > < \nu, q' > - < \nu, \eta' > < \nu, q >$$

holds a.e.-(t,x), where $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assune that the above identity holds outside a null set which is common to all η. We fix (t,x) at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course $supp.\nu \subset \Sigma$. Suppose that $supp.\nu \cap \{\rho > 0\} \neq \phi$. Let Σ_0 be the smallest triangle $\{z_0 \leq z \leq w \leq w_0\}$ such that $supp.\nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by P_0 the state (w_0, z_0). It will be verified that $\nu = \delta_{P_0}$. (the Dirac measure). First we show

Proposition 27

$$P_0 \in supp.\nu.$$

Proof. Suppose $P_0 \notin supp.\nu$. Since Σ_0 is the smallest triangle containing $supp.\nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with $supp.\nu \cap \{\rho > 0\}$. On neighborhoods of these intersection points we have

$$\begin{aligned} \eta^1 &\geq \frac{1}{M} e^{k(w_0-\epsilon)}, \\ \eta^2 &\geq \frac{1}{M} e^{-k(z_0+\epsilon)}. \end{aligned}$$

(See Proposition 24). Since ν, η^1, η^2 are nonnegative, we see

$$\begin{aligned} < \nu, \eta^1 > &\geq \frac{1}{M} e^{k(w_0-\epsilon)}, \\ < \nu, \eta^2 > &\geq \frac{1}{M} e^{-k(z_0+\epsilon)}. \end{aligned}$$

Since$P_0 \notin supp.\nu$, we have

$$< \nu, \eta^2 q^1 - \eta^1 q^2 > \leq M e^{k(w_0 - z_0 - \delta)}.$$

Taking $2\epsilon < \delta$, we have

$$\begin{aligned} \left|\frac{< \nu, q^1 >}{< \nu, \eta^1 >} - \frac{< \nu, q^2 >}{< \nu, \eta^2 >}\right| &= \left|\frac{< \nu, \eta^2 q^1 - \eta^1 q^2 >}{< \nu, \eta^1 >< \nu, \eta^2 >}\right| \\ &\leq M e^{-k(\delta - 2\epsilon)} \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Let β be a sufficiently small positive number, and we put

$$\begin{aligned} \Sigma_2 &= \{z_0 \leq z \leq w < w_0 - \beta\} \\ \Sigma_3 &= \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}. \end{aligned}$$

Then

$$\eta^1 e^{-kw} = (1 + O(1/c^2)) 2^N N! y^{N-1} (y + O(1/k))$$

is bounded on Σ_0 and we have

$$< \nu|_{\Sigma_2}, \eta^1 > \leq M e^{k(w_0 - \beta)}.$$

Taking $\epsilon = \beta/2$, we know

$$\frac{< \nu|_{\Sigma_2}, \eta^1 >}{< \nu, \eta^1 >} \leq M e^{-\beta k/2} \to 0.$$

Since $\partial\lambda_2/\partial w > 0$, we know

$$\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)$$

on Σ_3. Therefore we have

$$\begin{aligned} \frac{<\nu, q^1>}{<\nu, \eta^1>} &= \frac{<\nu|_{\Sigma_2}, \eta^1\lambda_2>}{<\nu, \eta^1>} + \frac{<\nu|_{\Sigma_3}, \eta^1\lambda_2>}{<\nu, \eta^1>} + \\ &+ O(1/k) \\ &\geq o(1) + \lambda_2(w_0 - \beta, z_0) \end{aligned}$$

Similarly we see

$$\frac{<\nu, q^2>}{<\nu, \eta^2>} \leq o(1) + \lambda_1(w_0, z_0 + \beta).$$

Therefore we have

$$\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \leq 0 + o(1).$$

Passing to the limit, we know

$$\lambda_2(w_0, z_0) \leq \lambda_1(w_0, z_0).$$

But this means $P_0 \in \{\rho = 0\}$, a contradiction. QED.

Let us fix a such that $z_0 < a < w_0$. We have

$$\begin{aligned} <\nu, B_n^3> &= <\nu, \eta^3><\nu, q_n^5> - <\nu, \eta_n^5><\nu, q^3>, \\ <\nu, B_n^4> &= <\nu, \eta^4><\nu, q_n^5> - <\nu, \eta_n^5><\nu, q^4>, \\ <\nu, \eta^3 q^4 - \eta^4 q^3> &= <\nu, \eta^3><\nu, q^4> - <\nu, \eta^4><\nu, q^3>, \\ <\nu, B_n> &= <\nu, \eta_n^5><\nu, q_n^6> - <\nu, \eta_n^6><\nu, q_n^5>. \end{aligned}$$

From (8.8) we know

$$<\nu, \eta^3 q^4 - \eta^4 q^3>> 0$$

and from(8.10) we know

$$<\nu, B_n^3> \to 0$$

Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

Proposition 28 *As $n \to \infty$, $<\nu, \eta_n^5>, <\nu, q_n^5>, <\nu, q_n^6>, <\nu, q_n^6>$ are bounded.*

Proposition 29 *As $n \to \infty$, we have $<\nu, B_n> \to 0$.*

Now, taking

$$\Phi_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

we put

$$\Phi(x) = \frac{1}{\beta}(\Phi_0(\frac{x+\beta}{\beta}) - \Phi_0(\frac{x-\beta}{\beta}))$$

for the generating function of η_n^5. Here $\beta = (1-\alpha)/2$. We put

$$\begin{aligned} S_+ &= \{z \le w, |w-a| \le \frac{1-3\alpha}{n}\}, \\ S_- &= \{z \le w, |z-a| \le \frac{1-3\alpha}{n}\}. \end{aligned}$$

Proposition 30 *As $n \to \infty$, we have*

$$< \nu|_{S_+}, ny^{2N} > + < \nu|_{S_-}, ny^{2N} > \to 0.$$

Proof. Put $S'_L = S_+ \cap S_L, S'_R = S_- \cap S_R$. It is sufficient to prove that

$$< \nu|_{S'_L}, ny^{2N} > + < \nu|_{S'_R}, ny^{2N} > \to 0.$$

From (8.11) we have

$$< \nu|_{S_L}, ny^{2N}A_1 + y^N A_2 > + < \nu|_{S_R}, ny^{2N}C_1 + y^N C_2 > \to 0.$$

Note

$$A_1 = (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2 \ge \frac{1}{M_0} > 0$$

on S'_L. Put

$$E_n = \{0 \le y \le (\frac{1}{n})^\mu\},$$

where μ is a positive parameter. Then $|y^N A_2| \le M(1/n)^{\mu N} = o(1)$ on $S_L \cap E_n$ and $|y^N A_2| \le Mny^{2N}(1/n)^{1-\mu N}$ on $S_L - E_n$. Choose $d_n \searrow 0$ such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha-d_n}^{1-\alpha} \Phi \ge (1/n)^{\mu_0}.$$

Then

$$(\int_{-1}^{H} \Phi)^2 \ge (1/n)^{2\mu_0}$$

for $|H| \le 1-\alpha-d_n$, and

$$|\Phi(H)| + |\int_{-1}^{H} \Phi| = o(1)$$

for $1-\alpha-d_n \le |H| \le 1$. Put

$$S_+^n = S_L \cap \{|w-a| \le \frac{1-\alpha-d_n}{n}\}.$$

Then $S'_L \subset S_+^n \subset S_L$ and

$$|y^N A_2| = o(1)$$

on $S_L - S_+^n$ and

$$\begin{aligned} ny^{2N}A_1 + y^N A_2 &\geq ny^{2N}(\frac{1}{M}(1/n)^{2\mu_0} - M(1/n)^{1-\mu N}) \\ &\geq 0 \end{aligned}$$

on $S_+^n - E_n$. Here we take $0 < 2\mu_0 < 1 - \mu N$. Then

$$\begin{aligned} < \nu|_{S_L}, ny^{2N}A_1 + y^N A_2 > &= < \nu|_{S_L \cap E_n}, ny^{2N}A_1 > + \\ &+ < \nu|_{S_L - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ o(1) \\ &\geq \frac{1}{M_0} < \nu|_{S_L' \cap E_n}, ny^{2N} > + \\ &+ < \nu|_{S_L - S_+^n \cap E_n}, ny^{2N}A_1 > + \\ &+ < \nu|_{S_L' - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ < \nu|_{S_+^n - S_L' - E_n}, ny^{2N}A_1 + y^N A_2 > + \\ &+ o(1) \\ &\geq \frac{1}{M_0} < \nu|_{S_L' \cap E_n}, ny^{2N} > + \\ &+ < \nu|_{S_L' - E_n}, ny^{2N}(\frac{1}{M_0} - M(1/n)^{1-\mu N} > + \\ &+ o(1) \\ &\geq \frac{1}{2M_0} < \nu|_{S_L'}, ny^{2N} > + \\ &+ o(1). \end{aligned}$$

Similarly we know

$$< \nu|_{S_R}, ny^{2N}C_1 + y^N C_2 > \geq \frac{1}{2M_0} < \nu|_{S_R'}, ny^{2N} > + o(1)$$

. Thus we see

$$< \nu|_{S_L'}, ny^{2N} > + < \nu|_{S_R'}, ny^{2N} > \to 0.$$

QED.

Proposition 31 *We have*

$$\nu|_{\{\rho > 0\}} = \delta_{P_0}.$$

Proof. Proposition 30 says that the projections $P_w\tilde{\nu}, P_z\tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N}\nu$ admits the Lebesgue lower derivatives which vanish at any a. Therefore we can claime that

$$supp.\nu \cap \{\rho > 0\} = \{P_0\}.$$

Since ν is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at P_0. Hence $C = 1$. QED.

Summing up we get the final

Theorem 2 *For any M_0 there is a positive number ϵ_0 such that if the initial data satisfy*

$$0 \le \rho_0(x) \le M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \le M_0.$$

and if $1/c^2 \le \epsilon_0$, then a subsequence of the approximate solutions U^Δ converges a.e. to a limit U which is a weak solution of the relativistic Euler equation.

11 Acknowledgment

The author expresses his sincere thanks to Dr. Tadayoshi Kano (Osaka University) and Pr. Hiroshi Uesaka (Nihon University) for helpfull informations on studies of the Euler-Poisson-Darboux equation. The thesis of Dr. Shigeharu Takeno was often reffered during the preparation of the manuscript. This work was, in part, financially supported by National Center for Theoretical Sciences Mathematics Division, R. O. C.

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